Four-Dimensional Yang–Mills Theory as a Deformation of Topological BF Theory

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Abstract

The classical action for pure Yang–Mills gauge theory can be formulated as a deformation of the topological BF theory where, beside the two-form field B, one has to add one extra-field η given by a one-form which transforms as the difference of two connections. The ensuing action functional gives a theory that is both classically and quantistically equivalent to the original Yang–Mills theory. In order to prove such an equivalence, it is shown that the dependency on the field η can be gauged away completely. This gives rise to a field theory that, for this reason, can be considered as semi-topological or topological in some but not all the fields of the theory. The symmetry group involved in this theory is an affine extension of the tangent gauge group acting on the tangent bundle of the space of connections. A mathematical

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analysis of this group action and of the relevant BRST complex is discussed in details.

1 Introduction

Among the open problems of quantum Yang–Mills (YM) theory, there is certainly the absence of any proof of the property of confinement, which is observed in nature for systems supposedly described by a YM Lagrangian, and which is proved true only in lattice formulations of the theory.

The non perturbative dynamics of gauge theories has been discussed at length in the literature from different point of views. More recently, some of the authors of this paper observed [8, 9] that the presence of a two-form field in the first-order formulation of YM theory might allow the construction of a surface observable that is related to ‘t Hooft's magnetic order operator [21]. A preliminary description of such surface observables can be found in [13, 7]; for a rigorous mathematical definition of these observables (in the case of paths of paths), s. [11].

The study of first-order YM theory was originally proposed in [17] as a way of taking into account strong-coupling effects (after some manipulations). For a discussion of this topic and its development, s. [19] and references therein.

Another aspect of first-order YM theory pointed out in [8] is its formal relationship with the topological BF theory [20, 5].[9] This suggested the possibility of finding Donaldson-like invariants [14] inside ordinary (not supersymmetric) YM theory with a mechanism similar to that described in [22].

The relation between YM and BF theory is actually more involved because the latter has less physical degrees of freedom than the former.

In this paper we establish the correct relation by using, as an intermediate step, a new theory—called BF-Yang–Mills (BFYM) theory—which contains a new one-form field η whose role is to provide the missing degrees of freedom.[10]

The first aim of this paper is to prove the classical and quantum equivalence between YM and BFYM theory. Cohomological proofs of this were considered in [18] and in [16]. In the present paper we give a different and more explicit proof by fixing the gauge of the theory in three different but equivalent ways, to which we refer as to the trivial, the covariant and the

[9] For the study of BF theory with observables, s. also [10].

[10] For an anticipation of some results of this paper by some of the authors, s. [16].

self-dual gauge fixing.

The reason for considering three different gauge fixings is that each of them provides a different setting for perturbation theory: In the trivial gauge, we have an expansion identical to that found in first-order YM theory (s. [15]). In the covariant gauge, the perturbative expansion around a flat connection can be organized by using the same propagators as in the topological pure BF theory (in the same gauge). Finally, the perturbative expansion in the self-dual gauge in a neighborhood of anti-self-dual connections makes use of the propagators of the topological BF theory with a cosmological term (in the same gauge).

The relation between BFYM and the BF theories is then discussed in a more formal way in the subsequent section by using the Batalin–Vilkovisky (BV) formalism [2] (which is a generalization of the more familiar BRST formalism [3]). In particular, we show that, after a *canonical transformation*, it is possible to perform safely the limit in which the YM coupling constant vanishes and obtain the pure BF theory plus the (covariant) kinetic term for the extra field η.

In the last part of the paper the geometrical aspects of BFYM are discussed. The group of symmetries of the BFYM theory (for an extended discussion, s. [12]) turns out to be an affine extension of the tangent gauge group. The action of this group on the space of fields is not free, but a BRST complex is obtainable directly from the action of the tangent gauge group on the space of fields of the theory.

The situation has both similarities and differences with respect to the case of topological gauge theories [4]. As in [4], the BRST equations are obtained as structure equations and Bianchi identities for the curvature of a suitable connection on the space of fields; but, differently from [4], the only symmetry for the connection A is the gauge invariance, as in the YM theory. Hence the BFYM theory can be seen as "semi-topological" (or topological in the field η and non-topological in the field A).

2 Preliminaries

In this section we introduce YM theory both in its usual second-order and in its first-order formulation. We prove the equivalence of the two formulations and discuss the problems related to the weak-coupling perturbative expansion

of the latter.

Then we will raise the issue of the topological embedding for a gauge field theory and discuss some of its general properties. It will be the aim of the following sections to prove that, through the topological embedding, it is possible to define a weak-coupling perturbative expansion of first-order YM theory around the topological BF theory.

2.1 Second-order YM theory

Let $P \to M$ be a principal G-bundle. The manifold M is a closed, simply connected, oriented four-manifold and G is $SU(N)$ or, more generally, a simple compact Lie group. The standard (second-order) YM action is the following local functional

$$S_{\mathrm{YM}}[A] = \frac{1}{4g_{\mathrm{YM}}^2} \langle F_A \, , \, F_A \rangle \, , \tag{1}$$

where g_{YM} is a real parameter (known as the YM coupling constant), and F_A is the curvature of the connection A. Here we consider the inner product $\langle \cdot \, , \, \cdot \rangle$ defined on the space $\Omega^*(M, \, \mathrm{ad}P)$ of forms on M with values in the adjoint bundle $\mathrm{ad}P$ and given by

$$\langle \alpha \, , \, \beta \rangle = \int_M \mathrm{Tr} \, (\alpha \wedge * \beta), \tag{2}$$

where $*$ is the Hodge operator on M. Even though the physical space–time is Minkowskian, we assume we have performed a Wick rotation so that M is a Riemannian manifold; viz., it has a *Euclidean* structure.

The gauge group (or group of gauge transformations) is, by definition, the group $\mathcal{G}$ of maps $g : P \to G$ which are equivariant, i.e., which satisfy the equation $g(ph) = \mathrm{Ad}_{h^{-1}} g(p)$ for any $p \in P$ and $h \in G$. Locally elements of $\mathcal{G}$ are represented by maps $M \to G$. Under a gauge trasformation $g \in \mathcal{G}$, the connection A transforms as

$$A \to A^g = \mathrm{Ad}_{g^{-1}} A + g^{-1} dg, \tag{3}$$

while any form $\psi \in \Omega^*(M, \, \mathrm{ad}P)$ transforms as

$$\psi \to \mathrm{Ad}_{g^{-1}} \psi.$$

The Yang–Mills action is gauge-invariant, i.e., is invariant under the action of $\mathcal{G}$. We denote the space of all connections by the symbol $\mathcal{A}$. With some restrictions, the group $\mathcal{G}$ acts freely on $\mathcal{A}$ and $\mathcal{A} \to (\mathcal{A}/\mathcal{G})$ is a principal $\mathcal{G}$-bundle.[11]

2.2 Classical analysis

A classical solution is a minimum of the action (modulo gauge transformations), i.e., a solution of the equation

$$d_A^* F_A = 0. \tag{4}$$

If we add a source $J(A)$ to the action, then the equations of motion become $d_A^* F_A + 2g_{\mathrm{YM}}^2 J' = 0$.

A particular class of solutions, is given by the (anti)self-dual connections, i.e., connections whose curvature is (anti)self-dual (i.e., satisfies the equation $F_A = \pm * F_A$).

We will denote by $\mathcal{M}_{\mathrm{YM}}$ the moduli space of solutions of the YM equations of motion modulo gauge transformations.

2.3 Quantum analysis

If we denote by $\mathcal{A}$ the space of connections, then the partition function is defined as

$$Z_{\mathrm{YM}} = \int_{\mathcal{A}/\mathcal{G}} \exp(-S_{\mathrm{YM}}). \tag{5}$$

The way physicists deal with this quotient in the quantum analysis is by introducing the BRST complex

$$\begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline 1 & & A & \\ 0 & \bar{c} & h_c & c \end{array} \tag{6}$$

(where each row has the same form-degree and each column has the same ghost-number), and the BRST transformations:

$$sA = d_A c, \quad sc = -\frac{1}{2}[c, c], \tag{7}$$

[11]For instance we can consider only the space of irreducible connections and the action of the group obtained by dividing $\mathcal{G}$ by its center. But in order to avoid cumbersome notations we will keep on writing the quotient as $\mathcal{A}/\mathcal{G}$.

$$s\bar{c} = h_c, \quad sh_c = 0. \tag{8}$$

Notice that the first equation is just the infinitesimal version of the action (3) of $\mathcal{G}$ on $\mathcal{A}$, while the second is the Maurer–Cartan equation for the group $\mathcal{G}$.

A (local) section $\mathcal{A}/\mathcal{G} \to \mathcal{A}$ is chosen by introducing a gauge fixing, i.e., by imposing a condition like, e.g., $d^*_{A_0}(A - A_0) = 0$ for A belonging to a suitable neighborhood of A_0. Correspondingly, one introduces a gauge-fixing fermion Ψ_{YM}, i.e., a local functional of ghost number -1 given by[12]

$$\Psi_{\mathrm{YM}} = \left\langle \bar{c}\,,\, d^*_{A_0}(A - A_0) \right\rangle, \tag{9}$$

where A_0 is a background connection. In perturbative calculations, we work in a neighborhood of a critical solution; i.e., we assume that A_0 is a solution of the classical equations of motion.

The original action is then replaced by

$$S^{\mathrm{g.f.}}_{\mathrm{YM}} = S_{\mathrm{YM}} + is\Psi_{\mathrm{YM}}, \tag{10}$$

and the functional integration is performed over the vector spaces to which the fields of the BRST complex belong (notice that the integration over the affine space of connections is replaced by an integration over the vector space $\Omega^1(M, \mathrm{ad}P)$ to which $A - A_0$ belongs).

To perform computations, it is also useful to assign each field a canonical (scaling) dimension so that the gauge-fixed action has dimension zero. Since the derivative, the volume integration and the BRST operator have respectively dimension 1, -4 and 0, we get the following table:

Dimension 0: c.

Dimension 1: A.

Dimension 2: $\bar{c}$, h_c.

Notice that the coupling constant g_{YM} has dimension 0.

[12]This way of implementing a gauge fixing is known as the *Landau gauge*. A more general gauge-fixing fermion implementing the same conditions is obtained by the replacement $\Psi_{\mathrm{YM}} \to \Psi_{\mathrm{YM}} + \lambda \langle \bar{c}\,,\, h_c \rangle$, where λ is a free parameter. By integrating out h_c and setting $\lambda = 1$, one recovers the *Feynman gauge*.

In this paper we will consider only Landau gauges.

Perturbative expansion The perturbative expansion of YM theory around a critical connection A_0 is performed by setting

$$A = A_0 + \sqrt{2}\, g_{\mathrm{YM}}\, \alpha. \tag{11}$$

By also using the gauge fixing condition, the quadratic part of the action then reads

$$\frac{1}{2}\left\langle \alpha\,,\, \check{\Delta}_{A_0}\alpha \right\rangle + i\left\langle h_c\,,\, d^*_{A_0}\alpha \right\rangle - i\left\langle \overline{c}\,,\, \Delta_{A_0} c\right\rangle, \tag{12}$$

where

$$\check{\Delta}_{A_0} = \Delta_{A_0} + *[*F_{A_0},\ ], \tag{13}$$

and Δ_{A_0} is the covariant Laplace operator. The $\alpha\alpha$-propagator is given by the inverse of $\check{\Delta}_{A_0}$.

2.4 The first-order formulation of YM theory

Now we consider the local functional

$$S_{\mathrm{YM}'}[A, E] = i\left\langle E\,,\, *F_A\right\rangle + g^2_{\mathrm{YM}}\left\langle E\,,\, E\right\rangle, \tag{14}$$

where F_A is the curvature of the connection A and $E \in \mathcal{B} \equiv \Omega^2(M, \mathrm{ad}P)$. As for the canonical dimension, E is assigned dimension 2.

The first-order YM theory—which we will prove in a moment to be equivalent to YM theory both at the classical and at the quantum level—is particularly interesting because of the new independent field E which allows the introduction of new observables which depend on loops of paths on M (or on the spanned surfaces) and could not be defined in ordinary YM theory [11].

The only symmetry of the theory corresponding to (14) is the gauge symmetry. It acts on the space of fields $\mathcal{A} \times \mathcal{B}$ as in (3) plus

$$E \to \mathrm{Ad}_{g^{-1}} E. \tag{15}$$

The group $\mathcal{G}$ acts freely on the manifold $\mathcal{A} \times \mathcal{B}$, which becomes a principal $\mathcal{G}$-bundle with the projection $(\mathcal{A} \times \mathcal{B}) \to \mathcal{A}$ being a bundle-morphism. As a consequence, the gauge fixing on A is enough to completely fix the first-order formulation of YM theory.

Remark The presence of an i in the action (14) may look odd but is necessary since the EF term is not positive definite. Notice that without the Wick rotation (thus on a Minkowskian manifold), the factor i would disappear from the action.

2.4.1 Classical equivalence

The critical points (which are not minima) of the action (14) correspond to the solutions of the following equations of motion:

$$\begin{array}{rcl} i * F_A + 2g_{\mathrm{YM}}^2 E & = & 0, \\ i * d_A E & = & 0. \end{array} \tag{16}$$

By applying the operator $*d_A$ to the first equation, we see that, because of the second equation, A must solve the YM equation (4). By the first equation we then see that E must be equal to $-(i/2g_{\mathrm{YM}}^2) * F_A$. Thus, the space of solutions of first-order YM theory is in a one-to-one correspondence with the space of solutions of second-order YM theory. Moreover, this correspondence is preserved by gauge equivalence. Therefore, the moduli spaces of the two theories are the same.

The presence of an i is a bit disturbing since it requires an imaginary solution for E. Moreover, it could seem that the factor i does not play any role in the classical equations. However, if we add a source $J(A)$ to the action (14), the second equation is replaced by $i * d_A E + J' = 0$. The application of $*d_A$ to the first equation gives the correct answer $d_A^* F_A + 2g_{\mathrm{YM}}^2 J' = 0$ just because of the i factor. Observe that if we were working on a Minkowski space, then we would get the correct answer by removing the factor i (this is because $*^2$ keeps track of the signature of the metric).

2.4.2 Quantum equivalence

It is not difficult to see that a Gaussian integration over E yields

$$\int_{(\mathcal{A}\times\mathcal{B})/\mathcal{G}} \exp(-S_{\mathrm{YM}'}[A,E])\ \mathcal{O}[A] \propto \int_{\mathcal{A}/\mathcal{G}} \exp(-S_{\mathrm{YM}}[A])\ \mathcal{O}[A], \tag{17}$$

where $\mathcal{O}$ is any gauge-invariant observable for YM theory.

Notice that the proportionality constant depends on g_{YM} because of the determinant coming from the E-integration. This dependence can be removed if one defines the functional measure for E as already containing this

factor. Anyhow, this constant factor is irrelevant since computing vacuum expectation values involves a ratio, and we will not take care of it.

More explicitly, the integration can be performed by introducing the BRST complex and the BRST transformations (7) and (8) plus

$$sE = [E, c], \tag{18}$$

which corresponds to the infinitesimal version of the gauge transformation (15).

Notice again that the presence of the i factor in the action is essential to make the Gaussian integration meaningful and to get the correct answer [instead of $\exp(+S_{\mathrm{YM}})$].

Finally, it is important to notice that $\mathcal{B}$ is a vector space, so *it is not necessary to fix a background solution E_0 to perform the integration; therefore, the equivalence between first- and second-order YM theories is non-perturbative.* However, one might also decide to fix a background solutions A_0 of YM equations and a background field $E_0 = -(i/2g_{\mathrm{YM}}^2) * F_{A_0}$ and integrate in the variable $E - E_0$; the result will be an equivalence with YM theory expanded around the same A_0.

2.4.3 The perturbative expansion

The exact computation of a functional integral is a formidable task. Usually one considers a perturbative expansion around a classical solution. YM theory can be computed as an asymptotic series in g_{YM} (i.e., in weak coupling) after defining the integration variable as $\alpha = (A - A_0)/(\sqrt{2}\, g_{\mathrm{YM}})$. In the first-order formulation, perturbation theory requires choosing a background for E as well and introducing the integration variable $\beta = \sqrt{2}\, g_{\mathrm{YM}}\, (E - E_0)$. The action will then contain the quadratic part

$$\langle \beta\,,\, *d_{A_0}\alpha \rangle + \left\langle 2g_{\mathrm{YM}}^2 E_0\,,\, *(\alpha \wedge \alpha) \right\rangle + \frac{1}{2}\,\langle \beta\,,\, \beta \rangle + \text{gauge fixing}$$

plus terms of order g_{YM} [notice that $g_{\mathrm{YM}}^2 E_0 = O(1)$]. From the quadratic part one reads the propagators and the Feynman rules leading to the usual ultraviolet behaviour [15].

Another possibility, which is interesting exploring, is to consider the term $\langle \beta\,,\, \beta \rangle$ as a perturbation. In this way the propagators will resemble those of

the topological pure BF theory defined by the action

$$S_{BF} = i \langle B \, , \, *F_A \rangle \, , \tag{19}$$

where the field B is just a new name for our E. Unfortunately, however, the operator acting on (α, β) in this scheme is not invertible since its kernel includes any pair $(0, \beta)$ with $\beta \in \ker(d_{A_0})$. In the pure BF theory there is no problem since the theory itself has a larger set of symmetries (s. Sec. 4) and, therefore, an additional gauge fixing is required.

Another way of seeing our problem is the following: The pure BF theory appears formally as the $g_{\mathrm{YM}}^2 \to 0$ limit of the first-order YM theory (after renaming E as B). However, the number of degrees of freedom are reduced in this limit and this shows that the limit is ill-defined.

The purpose of the next sections is to show that is possible to restore the missing degrees of freedom by introducing an extra field, and that this makes the above limit meaningful. The mechanism that will allow us to do so is the so-called "topological embedding."

Remark Notice that in the first-order YM theory one can define also a strong-coupling expansion (i.e., an asymptotic expansion in $1/g_{\mathrm{YM}}$) after integrating out the connection in (14) (notice that this integration is Gaussian). For details, s. [17, 19].

2.5 The "topological embedding"

The so-called topological embedding refers to the idea of "embedding" a topological into a physical theory.

The way we discuss such a scheme is partly related to the arguments presented in [1]. The basic idea is to consider an action $S[A]$ (or, more generally, an action $S[A, E]$) as a functional of an auxiliary field η as well. One then writes $S[A, \eta]$, but it is understood that $\delta S/\delta\eta = 0$.

Of course, this gives the theory a huge set of symmetries, viz., all possible shifts of η, so that η has no physical degrees of freedom. This is similar to what happens in the topological field theories of the so-called cohomological (or Witten) type where all fields are subject to such a symmetry. One might also speak of semi-topological theory since it is topological, in the previous sense, only in some field directions.

In our case the new field η belongs to $\Omega^1(M, \mathrm{ad}P)$, so that the pair (A, η) is an element of the tangent bundle $T\mathcal{A}$. The Lie group $T\mathcal{G}$, which can be represented as the semi-direct product of $\mathcal{G}$ and the abelian group $\Omega^0(M, \mathrm{ad}P)$, has a natural action on $T\mathcal{A}$ given by

$$(A, \eta) \to (A^g, \mathrm{Ad}_{g^{-1}}\eta + d_{A^g}\zeta), \tag{20}$$

where A^g is defined in (3) and $(g, \zeta) \in T\mathcal{G}$ [here $\zeta \in \Omega^0(M, \mathrm{ad}P)$].

It is convenient to combine the $T\mathcal{G}$-transformation with the translations acting on the field η under which the theory is invariant; viz., we write the transformation of (A, η) as

$$(A, \eta) \to (A^g, \mathrm{Ad}_{g^{-1}}\eta + d_{A^g}\zeta - \tau)\,, \tag{21}$$

where $\tau \in \Omega^1(M, \mathrm{ad}P)$ represents the translation. In this way we can write the group of the symmetries of the theory as the semidirect product of $T\mathcal{G}$ with $\Omega^1(M, \mathrm{ad}P)$. In the following we will denote this group by $\mathcal{G}_{\mathrm{aff}}$.

Unfortunately, the action of $\mathcal{G}_{\mathrm{aff}}$ given by (21) is not free. However, as will be shown in detail in subsec. 5.4, this problem can be successfully dealt with by considering the BRST complex defined by

$$s\eta = [\eta, c] + d_A\xi - \widetilde{\psi}, \tag{22}$$

where ξ and $\tilde{\psi}$ are new ghosts. For the BRST operator to be nilpotent they must obey the transformation rules

$$s\xi = -[\xi, c] + \widetilde{\phi}, \quad s\widetilde{\psi} = -[\widetilde{\psi}, c] + d_A\widetilde{\phi}, \tag{23}$$

where $\widetilde{\phi}$ is a ghost-for-ghost (i.e., has ghost number 2) which transforms as

$$s\widetilde{\phi} = [\widetilde{\phi}, c]. \tag{24}$$

For further details on the space $T\mathcal{A}$, on its symmetries and on the implementation of the BRST procedure, s. Sec. 5 and Ref. [12].

The quantization of such a theory requires a gauge fixing for this topological symmetry as well. The apparently trivial operation of adding a new field on which the theory does not depend and then gauging it away can have some interesting consequences:

1. A trivial gauge fixing for η (i.e., setting $\eta = 0$) is always available but if a non-trivial gauge fixing for η is chosen, this may introduce a non-trivial measure on the moduli space. Heuristically we have

$$\int_{T\mathcal{A}/\mathcal{G}_{\rm aff}} \exp(-S[A]) = \int_{\mathcal{A}/\mathcal{G}} \exp(-S[A])\ \mu[A, M], \qquad (25)$$

where[13] μ is the outcome of the η-integration and depends on A and on M and its metric structure through the chosen gauge fixing. Since one expects quantization not to depend on small deformations of the gauge fixing (in particular on small deformations of the metric of M), one can argue that $\mu[A, M]$ is a (possibly trivial) measure which, once integrated over the space of critical solutions modulo gauge-transformations (moduli space), gives a smooth invariant of M.

In particular we may choose as non-trivial gauge fixing an incomplete one which leaves only a finite number of symmetries, and then introduce a "topological observable" as a volume form.[14]

[13] The quotient space $T\mathcal{A}/\mathcal{G}_{\rm aff}$ is not a manifold since, as we discussed before, the action of $\mathcal{G}_{\rm aff}$ is not free. However, a BRST structure and the relevant quantization for the $\mathcal{G}_{\rm aff}$ symmetry is available. This is the reason why we can still write, at the heuristic level, the identity (25) between functional integrals.

[14] These are essentially the motivations of the approach of [1] where the following version of the YM action

$$S[A_0, A_q] = \frac{1}{4g_{\rm YM}^2} \left\langle F_{A_0 + g_{\rm YM} A_q}\,,\, F_{A_0 + g_{\rm YM} A_q} \right\rangle$$

is considered. This is equivalent to choosing $S[A, \eta] = (1/4g_{\rm YM}^2)\,\langle F_A\,,\, F_A\rangle$ by the change of variable (with constant Jacobian)

$$A_0 = A - \eta, \quad A_q = \frac{1}{g_{\rm YM}}\,\eta.$$

Moreover, if one also considers the following change of ghost variables (with constant Jacobian)

$$\begin{array}{rclrcl} C_0 & = & c - \xi, & C_q & = & \frac{1}{g_{\rm YM}}\,\xi, \\ \psi_0 & = & \widetilde{\psi} - [\eta, \xi], & \phi_0 & = & \frac{1}{2}[\xi, \xi] - \widetilde{\phi}, \end{array}$$

then the BRST transformations (7), (22), (23) and (24) become

$$\begin{array}{rclrcl} sA_0 & = & \psi_0 + d_{A_0} C_0, & sC_0 & = & \phi_0 - \frac{1}{2}[C_0, C_0], \\ s\psi_0 & = & -d_{A_0}\phi_0 - [\psi_0, C_0], & s\phi_0 & = & [\phi_0, C_0], \\ sA_q & = & -\frac{1}{g_{\rm YM}}\psi_0 + d_{A_0 + g_{\rm YM} A_q} C_q + [A_q, C_0], & sC_q & = & -\frac{1}{g_{\rm YM}}\phi_0 - [C_0, C_q] - \frac{g_{\rm YM}}{2}[C_q, C_q] \end{array}$$

2. When this procedure is applied the first-order formulation of YM theory, the limit $g^2_{\rm YM} \to 0$ becomes meaningful, as we will see in the next sections. Moreover, a weak-coupling perturbation theory with the propagators of one of the topological BF theories becomes available.

Before discussing the last point, it is better to rewrite the action $S_{\rm YM'}[A, E, \eta]$ of (extended) first-order YM theory by making the change of variables

$$B = E + \frac{1}{\sqrt{2}\, g_{\rm YM}}\, d_A\eta. \tag{26}$$

This yields the action

$$S_{\rm BFYM}[A, B, \eta] = i\,\langle B\,,\, *F_A\rangle + g^2_{\rm YM}\left\langle B - \frac{1}{\sqrt{2}\, g_{\rm YM}}\, d_A\eta\,,\, B - \frac{1}{\sqrt{2}\, g_{\rm YM}}\, d_A\eta\right\rangle, \tag{27}$$

which we will call BFYM theory since it is related to both the YM theory, after integrating out B and η, and to the BF theory, in the limit $g^2_{\rm YM} \to 0$; in particular, this limit yields

$$S_{\rm BFYM}[A, B, \eta] \overset{g^2_{\rm YM}\to 0}{\sim} i\,\langle B\,,\, *F_A\rangle + \frac{1}{2}\,\langle d_A\eta\,,\, d_A\eta\rangle\,, \tag{28}$$

where we recognize the BF theory plus a non-topological term that restores the degrees of freedom of YM theory. Notice that the presence of $1/(\sqrt{2}\, g_{\rm YM})$ in (26) is designed so as to give the kinetic term for η the correct normalization.

In the next section we will reconsider the equivalence of the BFYM and YM theories and prove it explicitly by using three different gauge fixings.

In section 4 we will discuss the limit $g^2_{\rm YM} \to 0$ and show that it is well-defined in the present context.

3 The BFYM theory

In this section we discuss the theory described by the action (27), i.e.,

$$S_{\rm BFYM}[A, B, \eta] = i\,\langle B\,,\, *F_A\rangle + g^2_{\rm YM}\left\langle B - \frac{1}{\sqrt{2}\, g_{\rm YM}}\, d_A\eta\,,\, B - \frac{1}{\sqrt{2}\, g_{\rm YM}}\, d_A\eta\right\rangle.$$

(which correspond to the BRST trasformation listed in [1]). In this way one recognizes the topological set of transformations for A_0 which is later reinterpreted as the background connection.

First we consider the equations of motion. They can be obtained directly by looking at the stationary points of (27) or from the equations of motion (16) of the first-order YM theory together with the change of variables (26). In any case, the equations of motion can be written, after a little algebra, as

$$\begin{array}{rcl} i * F_A + 2g_{\rm YM}^2 B - \sqrt{2}\, g_{\rm YM}\, d_A\eta &=& 0, \\ d_A^* F_A &=& 0, \\ d_A F_A &=& 0. \end{array} \tag{29}$$

Notice that the third equation is just the Bianchi identity.

Then we come to the symmetries. They can be obtained starting from the symmetries (3) and (15) of the first-order YM theory and from the topological symmetry (21) for η together with the change of variables (26). Explicitly, we have

$$\begin{array}{rcl} A &\rightarrow& A^g = {\rm Ad}_{g^{-1}} A + g^{-1} dg, \\ \eta &\rightarrow& {\rm Ad}_{g^{-1}}\eta + d_{A^g}\zeta - \sqrt{2}\, g_{\rm YM}\, \tau, \\ B &\rightarrow& {\rm Ad}_{g^{-1}} B + \frac{1}{\sqrt{2}\, g_{\rm YM}}\, [F_{A^g}, \zeta] - d_{A^g}\tau, \end{array} \tag{30}$$

with $(g, \zeta, \tau) \in \mathcal{G}_{\rm aff}$. The action of $\mathcal{G}_{\rm aff}$ on the space of triples (A, η, B) is again not a free action, but again, as it will be shown in a moment, a BRST complex and the relevant quantization are available.

Notice that we have rescaled $\tau \rightarrow \sqrt{2}\, g_{\rm YM}\, \tau$ so as to see the shift on η as a perturbation of the tangent group action; this is consistent with the limit $g_{\rm YM}^2 \rightarrow 0$ in (28). We will discuss this issue better in the next section, where we also explain why the B-transformation becomes singular in this limit. Notice that for the computations considered in this section all these rescalings are irrelevant.

A further remark concerns the geometric interpretation of the field B: since it transforms as $d_A\eta$, it is natural to see it as an element of a the tangent space $T_{F_A}\mathcal{B}$ and not of $\mathcal{B}$.

To quantize the theory we have to describe the BRST symmetry. Again the BRST transformations for BFYM theory can be obtained from (7), (18), (22), (23), (24) and (26). Explicitly they read

$$\begin{array}{rclcrcl} sA &=& d_A c, && sc &=& -\frac{1}{2}[c, c], \\ s\eta &=& [\eta, c] + d_A\xi - \sqrt{2}\, g_{\rm YM}\, \widetilde{\psi}, && s\xi &=& -[\xi, c] + \sqrt{2}\, g_{\rm YM}\, \widetilde{\phi}, \\ sB &=& [B, c] + \frac{1}{\sqrt{2}\, g_{\rm YM}}\, [F_A, \xi] - d_A\widetilde{\psi}, && && \\ s\widetilde{\psi} &=& -[\widetilde{\psi}, c] + d_A\widetilde{\phi}, && s\widetilde{\phi} &=& [\widetilde{\phi}, c]. \end{array} \tag{31}$$

where c and ξ have ghost number 1 and belong to $\Omega^0(M, \mathrm{ad}P)$, $\widetilde{\psi} \in \Omega^1(M, \mathrm{ad}P)$ and has ghost number 1, and $\widetilde{\phi} \in \Omega^0(M, \mathrm{ad}P)$ and has ghost number 2. Notice that we have rescaled $(\widetilde{\psi}, \widetilde{\phi}) \to \sqrt{2}\, g_{\mathrm{YM}}(\widetilde{\psi}, \widetilde{\phi})$ so as to see the shifts as perturbations of the $T\mathcal{G}$ transformations on η and ξ.

To study the theory, both at the classical and at the quantum level, we have to fix the symmetries (30). After having done this, our first aim will be to prove that the gauge-fixed BFYM and YM theories are classically equivalent, i.e., that their moduli spaces are in one-to-one correspondence with each other. Our second aim will be to prove the quantum equivalence, i.e.,

$$\int_{(T\mathcal{A}\times T_{F_A}\mathcal{B})/\mathcal{G}_{\mathrm{aff}}} \exp(-S_{\mathrm{BFYM}}[A,\eta,B])\ \mathcal{O}[A] \propto \int_{\mathcal{A}/\mathcal{G}} \exp(-S_{\mathrm{YM}}[A])\ \mathcal{O}[A]. \tag{32}$$

As in (17), the proportionality constant will depend on g_{YM} but will not affect the vacuum expectation values, and we will not take care of it.

Notice that, as it was for the case in (25), the quotient $(T\mathcal{A}\times T_{F_A}\mathcal{B})/\mathcal{G}_{\mathrm{aff}}$ is not a manifold since the action of $\mathcal{G}_{\mathrm{aff}}$ is not free. The same argument of footnote 13 applies here and a detailed discussion on how to deal with the non-freedom of these group action is considered in subsec. 5.5.

The formal computation of the functional integral can be performed after choosing a gauge fixing. In general, whenever we verify that some conditions are a gauge fixing (at least in a neighborhood of critical solutions), we expect the equivalence to be realized (in that neighborhood); for we can always go back to the variables A, η, E by (26) and perform the Gaussian E-integration. The η-integration should give at most some topological contributions since η appears only in s-exact terms now. However, the change of variables (26) becomes singular as $g_{\mathrm{YM}} \to 0$, so we prefer to work the equivalence out by using the variables A, η, B.

We will consider three different gauge fixings which we call the *trivial*, the *covariant* and the *self-dual* gauge fixings. The last two of them will be dealt with in the next two subsections, and the conditions under which classical and quantum equivalence are true will be discussed. As for the trivial gauge fixing, characterized by the condition $\eta = 0$ plus a gauge-fixing condition on A, we see immediately that BFYM theory turns out to be equivalent to the first-order formulation of YM theory which, as we proved in the previous section, is equivalent to the second-order formulation.

The other two gauge fixings are equivalent to the trivial one (when they are defined), so we can be sure of the equivalence between BFYM and YM theory in any of these gauges without any further computation. However, we prefer to check the equivalence explicitly, for this treatment also produces the correct framework to consider perturbation theory around BF theories.

Obviously a weak-coupling expansion as in first-order YM theory is always possible, and this is the only possibility in the trivial gauge. In the covariant gauge, however, we will show that perturbation theory around a flat connection can be organized in a different way so that the AB-sector and the η-sector of the theory decouple in the unperturbed action and the AB-propagator turns out to be the propagator of the topological pure BF theory (in the covariant gauge). Finally, in the self-dual gauge, perturbation theory around an anti-self-dual non-trivial connection (the only kind of connection around which this gauge is well defined) can again be organized in such a way that the AB-sector and the η-sector decouple in the unperturbed action; moreover, the propagators in the AB-sector are recognized as those of the topological BF theory with a cosmological term (in the self-dual gauge).

3.1 The covariant gauge fixing

The covariant gauge fixing, which will be discussed explicitly in subsec. 5.6, is characterized by a gauge-fixing condition on A together with

$$d_A^*\eta = 0, \quad \eta \perp \mathrm{Harm}_A^1(M,\, \mathrm{ad}P), \quad d_A^* B \in d_A \Omega^0(M,\, \mathrm{ad}P), \tag{33}$$

where

$$\mathrm{Harm}_A^k(M,\, \mathrm{ad}P) \equiv \{\omega \in \Omega^k(M,\, \mathrm{ad}P) \mid \Delta_A \omega = 0\}, \tag{34}$$

and

$$\Delta_A \equiv d_A^* d_A + d_A d_A^* : \Omega^*(M,\, \mathrm{ad}P) \to \Omega^*(M,\, \mathrm{ad}P). \tag{35}$$

Notice that if $b^1[A] = \dim\, \mathrm{Harm}_A^1(M,\, \mathrm{ad}P)$ is not constant on the whole space $\mathcal{A}$, the covariant gauge fixing is consistently defined only in those open regions where it is constant. In particular, we will denote by $\mathcal{N}$ the open neighborhood of the space of connections where this is true (in particular cases, $\mathcal{N}$ may be the whole $\mathcal{A}$).

By consistency, on the shift τ in (30) we must impose the same conditions as those which fix the $T\mathcal{G}$ symmetry on η, viz.,

$$d_A^*\tau = 0, \quad \tau \perp \mathrm{Harm}_A^1(M,\, \mathrm{ad}P).$$

Similarly, in the context of BRST quantization, the ghost $\widetilde{\psi}$ is subject to the same conditions

$$d_A^* \widetilde{\psi} = 0, \quad \widetilde{\psi} \perp \mathrm{Harm}^1_A(M,\, \mathrm{ad}P). \tag{36}$$

Since we have $\mathrm{Harm}^0_A(M,\, \mathrm{ad}P) = \{0\}$ (which is a consequence of taking A irreducible), this is actually a gauge fixing. Notice that there is a set of interpolating (complete) gauge fixings between (33) and the trivial gauge fixing, $\eta = 0$, which can also be written as

$$d_A^* \eta = 0, \quad \eta \in d_A \Omega^0(M,\, \mathrm{ad}P).$$

The interpolating gauge fixings are then given by

$$\lambda d_A^* B + (1-\lambda)\eta \in d_A \Omega^0(M,\, \mathrm{ad}P),$$

with $\lambda \in [0,1]$.

One might also choose λ to be smooth but not constant on $\mathcal{A}$. In particular, one could choose λ to be constant and equal to 1 in an open neighborhood of the space of critical connections contained in the neighborhood $\mathcal{N}$, and constant and equal to 0 outside $\mathcal{N}$. In this way one would obtain a gauge fixing that is defined on the whole space $\mathcal{A}$ and restricts to the covariant gauge fixing close to the critical connections.

3.1.1 Classical equivalence

Consider the equations of motion (29). The second and the third tell us that A is a solution of the YM equations. The first implies that

$$d_A^*(2g_{\mathrm{YM}}^2 B - \sqrt{2}\, g_{\mathrm{YM}}\, d_A \eta) = 0,$$

so that

$$\left\langle d_A \eta\, ,\, 2g_{\mathrm{YM}}^2 B - \sqrt{2}\, g_{\mathrm{YM}}\, d_A \eta \right\rangle = 0.$$

On the other hand, the gauge-fixing conditions (33) imply that

$$\langle d_A \eta\, ,\, B \rangle = 0.$$

So we conclude that

$$||d_A \eta||^2 = 0.$$

By the positivity of the norm (remember that we are in a Riemannian manifold) we get then $d_A\eta = 0$. Since the gauge fixing also imposes $d_A^*\eta = 0$ and requires η not to be harmonic, we conclude that

$$\eta = 0. \tag{37}$$

Finally, inserting this result in (29) yields

$$B = -\frac{i}{2g_{\mathrm{YM}}^2} * F_A. \tag{38}$$

Therefore, we have shown that a solution A of the YM equations completely determines a solution of BFYM equations in the covariant gauge fixing.

Notice that this solution coincides with that obtained with the trivial gauge fixing.

3.1.2 Quantum equivalence

To implement the covariant gauge fixing in the BRST formalism, we have first to introduce the full BRST complex which generalizes (6). It is useful to organize all the fields in the following tables where each row has the same form-degree and each column has the same ghost-number:

$$\begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline 1 & & (A,\eta) & \\ 0 & (\overline{c},\overline{\xi}) & (h_c, h_\xi) & (c,\xi) \end{array} \tag{39}$$

$$\begin{array}{c|ccccc} & -2 & -1 & 0 & 1 & 2 \\ \hline 2 & & & B & & \\ 1 & & \overline{\widetilde{\psi}} & h_{\widetilde{\psi}} & \widetilde{\psi} & \\ 0 & \overline{\widetilde{\phi}}_1 & h_{\widetilde{\phi}_1} & \overline{\widetilde{\phi}}_2 & h_{\widetilde{\phi}_2} & \widetilde{\phi} \end{array} \tag{40}$$

The BRST transformations are given by (31) together with

$$\begin{array}{rclrclrclrcl} s\overline{c} &=& h_c, & sh_c &=& 0, & s\overline{\xi} &=& h_\xi, & sh_\xi &=& 0, \\ s\overline{\widetilde{\psi}} &=& h_{\widetilde{\psi}}, & sh_{\widetilde{\psi}} &=& 0, & & & & & & \\ s\overline{\widetilde{\phi}}_1 &=& h_{\widetilde{\phi}_1}, & sh_{\widetilde{\phi}_1} &=& 0, & s\overline{\widetilde{\phi}}_2 &=& h_{\widetilde{\phi}_2}, & sh_{\widetilde{\phi}_2} &=& 0. \end{array} \tag{41}$$

If harmonic one-forms are present, in order to implement the covariant gauge fixing, (33) and (36), it is better to rewrite the BRST transformations for η and $\widetilde{\psi}$ displaying the harmonic contribution.

First we take an orthogonal basis $\omega_i[A]$ of $\mathrm{Harm}^1_A(M, \mathrm{ad}P)$, with $i = 1, \ldots, b^1[A] = \dim \mathrm{Harm}^1_A(M, \mathrm{ad}P)$. To be consistent with the scaling dimensions, we normalize this basis as

$$\langle \omega_i[A] \, , \, \omega_j[A] \rangle = \delta_{ij} \sqrt{V}, \tag{42}$$

where V is the volume of the manifold M.

As a consequence of the fact that $\omega_i[A^g] = \mathrm{Ad}_{g^{-1}} \omega_i[A]$, we get the BRST transformation rule

$$s\omega_i[A] = [\omega_i[A], c]. \tag{43}$$

Then we add a family of constant ghosts k^i and r^i (respectively of ghost number 1 and 2) and BRST transformation rules

$$sk^i = \sqrt{2}\, g_{\mathrm{YM}}\, r^i, \quad sr^i = 0. \tag{44}$$

Finally, we rewrite the BRST transformations for η and $\widetilde{\psi}$ as

$$\begin{array}{rcl} s\eta & = & [\eta, c] + d_A \xi - \sqrt{2}\, g_{\mathrm{YM}}\, \widetilde{\psi} + k^i \omega_i[A], \\ s\widetilde{\psi} & = & -[\widetilde{\psi}, c] + d_A \widetilde{\phi} + r^i \omega_i[A], \end{array} \tag{45}$$

where a sum over repeated indices is understood. It is easily verified that the BRST operator is still nilpotent.

To implement the gauge fixing, we have then to build the BRST complex, i.e., add to (39) and (40) the following table:

$$\begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \\ \hline & \overline{k}^i & h^i_k & k^i & \\ \overline{r_1}^i & h^i_{r_1} & \overline{r_2}^i & h^i_{r_2} & r^i \end{array} \tag{46}$$

where each column has the displayed ghost-number. We conclude by giving the last BRST transformations, viz.,

$$\begin{array}{rclrclrclrcl} s\overline{k}^i & = & h^i_k, & sh^i_k & = & 0, & & & & & & \\ s\overline{r_1}^i & = & h^i_{r_1}, & sh^i_{r_1} & = & 0, & s\overline{r_2}^i & = & h^i_{r_2}, & sh^i_{r_2} & = & 0. \end{array} \tag{47}$$

Now we are in a position to write down the gauge-fixing fermion that implements the conditions (33) and (36):

$$\begin{aligned}\Psi = \ & \Psi_{\mathrm{YM}} + \\ + \ & \left\langle \overline{\xi}\,,\, d_A^* \eta \right\rangle + \overline{k}^i \left\langle \omega_i[A]\,,\, \eta \right\rangle + \\ + \ & \left\langle \overline{\widetilde{\phi}_1}\,,\, d_A^* \widetilde{\psi} \right\rangle + \overline{r_1}^i \left\langle \omega_i[A]\,,\, \widetilde{\psi} \right\rangle + \\ + \ & \left\langle \overline{\widetilde{\psi}}\,,\, d_A^* B + d_A \overline{\widetilde{\phi}_2} + \overline{r_2}^i\, \omega_i[A] \right\rangle,\end{aligned} \tag{48}$$

where Ψ_{YM} is a gauge-fixing fermion for YM theory like, e.g., in (9). Notice that both $d_A^* B$ and $d_A \overline{\widetilde{\phi}_2}$ are in the orthogonal complement of $\mathrm{Harm}_A^1(M,\, \mathrm{ad}P)$; thus, to implement the second gauge-fixing condition in (33), we must take $\overline{\widetilde{\psi}}$ in this orthogonal complement as well. This is accomplished by the last term in (48).

The gauge-fixed action will then read

$$S_{\mathrm{BFYM}}^{\mathrm{g.f.}} = S_{\mathrm{BFYM}} + is\Psi. \tag{49}$$

Notice the double role played here by $\overline{\widetilde{\psi}}$, $\overline{\widetilde{\phi}_2}$ and $\overline{r_2}^i$: On the one hand, we can see $\overline{\widetilde{\psi}}$ as the antighost orthogonal to the harmonic forms that allows an explicit implementation of the gauge-fixing condition for B, viz.,

$$d_A^* B + d_A \overline{\widetilde{\phi}_2} = 0; \tag{50}$$

on the other hand, we can see $\overline{\widetilde{\phi}_2}$ and $\overline{r_2}^i$ as antighosts that implement on $\overline{\widetilde{\psi}}$ the same conditions as those satisfied by $\widetilde{\psi}$, viz.,

$$d_A^* \overline{\widetilde{\psi}} = 0, \quad \overline{\widetilde{\psi}} \perp \mathrm{Harm}_A^1(M,\, \mathrm{ad}P). \tag{51}$$

As in the case of YM theory, it is useful to assign a canonical dimension to all the fields in such a way that the gauge-fixed action, the derivative, the volume integration and the BRST operator have, respectively, dimensions 0, 1, -4 and 0. Therefore, we get

Dimension 0: $c, \xi, \widetilde{\phi}, k^i, r^i$.

Dimension 1: $A, \eta, \overline{\widetilde{\psi}}, h_{\widetilde{\psi}}, \widetilde{\psi}$.

Dimension 2: $B, \overline{c}, \overline{\xi}, h_c, h_\xi, \overline{\widetilde{\phi}_1}, h_{\widetilde{\phi}_1}, \overline{\widetilde{\phi}_2}, h_{\widetilde{\phi}_2}, \overline{k}^i, h_k^i, \overline{r_1}^i, h_{r_1}^i, \overline{r_2}^i, h_{r_2}^i$.

The explicit computation Our first task is to compute $s\Psi$. This will produce many terms which we can divide into two classes: terms that contain a Lagrange multiplier (the h-fields) and terms that do not. The former will impose the gauge-fixing conditions (33), (50), (36) and (51) (notice that—and this is the advantage of working in the Landau gauge—we do not have quadratic terms in the hs, so the h-integrations produce δ-functionals of the constraints). In the computation of the latter, several terms will be canceled after explicitly imposing these gauge-fixing conditions. In particular, all the terms containing the ghost c (apart from those in $s\Psi_{\mathrm{YM}}$) are killed since the covariant gauge-fixing conditions are $\mathcal{G}$-equivariant; e.g., in the s-variation of $\left\langle \overline{\xi}\,,\, d_A^*\eta \right\rangle$, we will remove the term $\left\langle \overline{\xi}\,,\, [d_A^*\eta, c] \right\rangle$ by imposing $d_A^*\eta = 0$. Particular care has to be taken in the variation of the last line in (48) since $\overline{\widetilde{\phi}}_2$ is not $\mathcal{G}$-equivariant; the c-dependent part will then read (by adding and subtracting $d_A[\overline{\widetilde{\phi}}_2, c]$)

$$\left\langle \overline{\widetilde{\psi}}\,,\, [d_A^* B + d_A \overline{\widetilde{\phi}}_2 + \overline{r_2}^i \omega_i[A], c] - d_A[\overline{\widetilde{\phi}}_2, c] \right\rangle .$$

The first term then vanishes by the gauge-fixing condition (50) of B, while the last term can be rewritten as $\left\langle d_A^* \overline{\widetilde{\psi}}\,,\, [\overline{\widetilde{\phi}}_2, c] \right\rangle$ and vanishes by the gauge-fixing conditions (51) of $\overline{\widetilde{\psi}}$.

By imposing the gauge-fixing conditions, we can also simplify the action S_{BFYM}: the effect is to eliminate the mixed term in B and η.

Finally, we see that, thanks to the gauge-fixing conditions, we can always replace $d_A^* d_A$ by the invertible operator Δ_A' defined as

$$\Delta_A' = \Delta_A + \pi_{\mathrm{Harm}_A} = \begin{cases} 1 & \text{on } \ker(\Delta_A) = \mathrm{Harm}_A \\ \Delta_A & \text{on } \mathrm{coker}(\Delta_A) \end{cases} \tag{52}$$

where π_{Harm_A} is the projection to harmonic forms. Notice that $\Delta_A' = \Delta_A$ on zero-forms since A is an irreducible connection. In the following, we will denote by G_A the inverse of Δ_A' and by $\det'(\Delta_A)$ the determinant of Δ_A'.

Therefore, the gauge-fixed action—after all these simplifications—reads

$$\begin{aligned}
S^{\text{cov. g.f.}}_{\text{BFYM}} &= i\left\langle B\,,\,*F_A\right\rangle + g^2_{\text{YM}}\left\langle B\,,\,B\right\rangle + \tfrac{1}{2}\left\langle \eta\,,\,\Delta'_A\eta\right\rangle + \\
&+ i\Big(s\Psi_{\text{YM}} + \left\langle h_\xi\,,\,d^*_A\eta\right\rangle + h^i_k\left\langle \omega_i[A]\,,\,\eta\right\rangle + \\
&+ \left\langle h_{\widetilde{\phi}_1}\,,\,d^*_A\widetilde{\psi}\right\rangle + h^i_{r_1}\left\langle \omega_i[A]\,,\,\widetilde{\psi}\right\rangle + \\
&+ \left\langle h_{\widetilde{\psi}}\,,\,d^*_A B + d_A\overline{\widetilde{\phi}_2} + \overline{r_2}^i\,\omega_i[A]\right\rangle + \\
&+ \left\langle h_{\widetilde{\phi}_2}\,,\,d^*_A\overline{\widetilde{\psi}}\right\rangle + h^i_{r_2}\left\langle \omega_i[A]\,,\,\overline{\widetilde{\psi}}\right\rangle + \\
&- \left\langle \overline{\xi}\,,\,\Delta_A\xi\right\rangle - \overline{k}^i k^j\delta_{ij}\sqrt{V} + \\
&+ \left\langle \overline{\widetilde{\phi}_1}\,,\,\Delta_A\widetilde{\phi}\right\rangle + \overline{r_1}^i r^j\delta_{ij}\sqrt{V} + \\
&- \tfrac{1}{\sqrt{2}g_{\text{YM}}}\left\langle d_A\overline{\widetilde{\psi}}\,,\,[F_A,\xi]\right\rangle + \left\langle \overline{\widetilde{\psi}}\,,\,\Delta'_A\widetilde{\psi}\right\rangle\Big).
\end{aligned}\tag{53}$$

Notice that there is only one term which is singular as $g_{\text{YM}} \to 0$. However, this singularity can be removed easily if one rescales $\xi \to g_{\text{YM}}\,\xi$ and $\overline{\xi} \to \overline{\xi}/g_{\text{YM}}$.

Now we can start integrating out fields in order to prove (32). We want to point out that it is not necessary to choose a background for η and B since they already belong to vector spaces.

Step 1 Integrate $\overline{k}^i, k^i, \overline{r_1}^i, r^i$.

The integration over the first two variables yields $V^{b^1[A]/2}$, while the integration over the last two of them yields $V^{-b^1[A]/2}$; therefore, the contributions cancel each other.

Step 2 Integrate $\overline{\xi}, \xi, \overline{\widetilde{\phi}_1}, \widetilde{\phi}$.

The first integration yields $\det\Delta^{(0)}_A$, while the second yields $(\det\Delta^{(0)}_A)^{-1}$ and they cancel each other. Notice that there are no sources in $\overline{\xi}$, so the integration kills the term in ξ.

Step 3 Integrate $\overline{\widetilde{\psi}}, \widetilde{\psi}, h_{\widetilde{\phi}_1}, h_{\widetilde{\phi}_2}, h^i_{r_1}, h^i_{r_2}$.

The integration over the first two fields yields $\det'\Delta^{(1)}_A$. Moreover, since there are linear sources in $\overline{\widetilde{\psi}}$ and $\widetilde{\psi}$, viz.,

$$i\left\langle d_A h_{\widetilde{\phi}_1} + h^i_{r_1}\omega_i[A]\,,\,\widetilde{\psi}\right\rangle - i\left\langle \overline{\widetilde{\psi}}\,,\,d_A h_{\widetilde{\phi}_2} + h^i_{r_2}\omega_i[A]\right\rangle,$$

the Gaussian integration will give the following contribution to the action

$$i\left(\left\langle h_{\widetilde{\phi}_1}\,,\, X_A h_{\widetilde{\phi}_2}\right\rangle + h^i_{r_1} h^j_{r_2}\delta_{ij}\sqrt{V}\right), \tag{54}$$

where

$$X_A = d_A^* G_A d_A : \Omega^0(M,\, \mathrm{ad}P) \to \Omega^0(M,\, \mathrm{ad}P). \tag{55}$$

Notice that there are no other terms since $G_A\omega_i[A] = \omega_i[A]$ and $d_A^*\omega_i[A] = 0$.

Now the integration over $h_{\widetilde{\phi}_1}$ and $h_{\widetilde{\phi}_2}$ yields $\det X_A$ (we will show shortly that X_A is invertible), while the integration over $h^i_{r_1}$ and $h^i_{r_2}$ yields $V^{b^1[A]/2}$.

Therefore, the net contribution of this step is given by

$$\det{}'\Delta^{(1)}_A\ \det X_A\, V^{b^1[A]/2}.$$

Step 4 Integrate η, h_ξ, h^i_k.

The first integration yields $(\det{}'\Delta^{(1)}_A)^{-1/2}$; moreover, the linear source in η, viz.,

$$i\left\langle d_A h_\xi + h^i_k \omega_i[A]\,,\, \eta\right\rangle,$$

produces the following contribution to the action:

$$\langle h_\xi\,,\, X_A h_\xi\rangle + h^i_k h^j_k \delta_{ij}\sqrt{V}.$$

Then the h_ξ-integration yields $(\det X_A)^{-1/2}$, while the h^i_k-integrations yield $(4V)^{-b^1[A]/4}$.

Therefore, the net contribution of this step is given by

$$(\det{}'\Delta^{(1)}_A\ \det X_A)^{-1/2}\,(4V)^{-b^1[A]/4}.$$

Step 5 Integrate B.

Apart from an irrelevant g_{YM}-dependent factor, the Gaussian B-integration with source

$$i\left\langle B\,,\, *F_A + d_A h_{\widetilde{\psi}}\right\rangle$$

gives the following contribution to the action

$$\frac{1}{4g^2_{\mathrm{YM}}}\left(\langle F_A\,,\, F_A\rangle + \left\langle h_{\widetilde{\psi}}\,,\, \Delta'_A h_{\widetilde{\psi}}\right\rangle\right), \tag{56}$$

the mixed terms disappearing because of the Bianchi identity.

Step 6 Integrate $h_{\widetilde{\psi}}, \overline{\widetilde{\phi}_2}, \overline{r_2}^i$.

The $h_{\widetilde{\psi}}$-integration with quadratic term given in (56) and source

$$i \left\langle h_{\widetilde{\psi}}\,,\, d_A \overline{\widetilde{\phi}_2} + \overline{r_2}^i \omega_i[A] \right\rangle$$

yields $(\det' \Delta_A^{(1)})^{-1/2}$ plus the contribution

$$g_{\mathrm{YM}}^2 \left(\left\langle \overline{\widetilde{\phi}_2}\,,\, X_A \overline{\widetilde{\phi}_2} \right\rangle + \overline{r_2}^i \overline{r_2}^j \delta_{ij} \sqrt{V} \right).$$

Then the remaining integrations yield $(\det X_A)^{-1/2}$ and $(4 g_{\mathrm{YM}} 4V)^{-b^1[A]/4}$.

Therefore, the net contribution of this step is

$$(\det' \Delta_A^{(1)})^{-1/2} \, (\det X_A)^{-1/2} \, (4 g_{\mathrm{YM}} 4V)^{-b^1[A]/4}.$$

The operator X_A is invertible In order to complete all the steps in the functional integration, we still have to prove that the operator X_A is invertible.

Let us represent the space $\Omega^1(M,\, \mathrm{ad}P)$ as the sum of three orthogonal subspaces: the vertical subspace $V_A = d_A \Omega^0(M,\, \mathrm{ad}P)$, $\mathrm{Harm}_A^1(M,\, \mathrm{ad}P)$, and $\hat{H}_A \equiv H_A \ominus \mathrm{Harm}_A^1(M,\, \mathrm{ad}P)$ where $H_A = \ker d_A^*$ is the horizontal subspace. Here horizontality and verticality are defined with respect to the connection form on $\mathcal{A}$ given by

$$G_A \, d_A^* : \Omega^1(M,\, \mathrm{ad}P) \to \Omega^0(M,\, \mathrm{ad}P).$$

The operator $\Delta_A^{(1)} : \hat{H}_A \oplus V_A \to \hat{H}_A \oplus V_A$ is injective and satisfies the following relation:

$$\left\langle \eta_1\,,\, \Delta_A^{(1)} \eta_2 \right\rangle = \langle (d_A + d_A^*) \eta_1\,,\, (d_A + d_A^*) \eta_2 \rangle\,.$$

Hence both $\Delta_A^{(1)}$ and its inverse G_A are (formally) self-adjoint and positive. We can consider the (formal) "square root" $G_A^{1/2}$ and have, for any $\zeta \in \Omega^0(M,\, \mathrm{ad}P)$,

$$\langle \zeta\,,\, X_A \zeta \rangle = \langle d_A \zeta\,,\, G_A d_A \zeta \rangle = ||G_A^{1/2} d_A \zeta||^2.$$

Hence X_A is invertible.

Conclusions Collecting all the contributions, we get YM as the effective action. All the determinants cancel each other and the only net contribution of all the integrations is a factor $(2g_{\rm YM})^{-b^1[A]}$. However, $b^1[A]$ is constant in the neighborhood $\mathcal{N}$. If this neighborhood does not coincide with $\mathcal{A}$, one can choose an interpolating gauge that smoothly connects the covariant gauge in the interior of $\mathcal{N}$ with the trivial gauge outside. This ends the proof of (32) in the covariant gauge.

3.1.3 The perturbative expansion

As we have remarked after eqn. (53), a suitable rescaling of ξ and $\overline{\xi}$ removes any singularity as $g_{\rm YM} \to 0$. This allows weak-coupling perturbation theory.

To start with, one has to consider the fluctuations α, e and β of the fields A, η, B; viz.,

$$\begin{array}{rcl} A & = & A_0 + q\alpha, \\ \eta & = & \eta_0 + e, \\ B & = & B_0 + \frac{1}{q}\beta, \end{array} \tag{57}$$

where q is a free parameter, and (A_0, η_0, B_0) is a critical point of the action: i.e., A_0 is a critical connection, $\eta_0 = 0$ and B_0 is given by (38). On the fluctuations, the covariant gauge fixing reads

$$\begin{array}{lcl} d^*_{A_0} e + O(q) & = & 0, \quad e \perp \mathrm{Harm}^1_A(M,\, \mathrm{ad}P), \\ d^*_{A_0}\beta + q^2 * [\alpha, *B_0] + O(q) & \in & d_A\Omega^0(M,\, \mathrm{ad}P) + \mathrm{Harm}^1_A(M,\, \mathrm{ad}P). \end{array} \tag{58}$$

[Recall that $q^2 B_0 = O(1)$.]

The general case The quadratic part of the gauge-fixed action (53) reads

$$i\,\langle \beta\,,\, *d_{A_0}\alpha\rangle + \frac{1}{2}\left(\frac{q}{g_{\rm YM}}\right)^2 \langle F_{A_0}\,,\, \alpha\wedge\alpha\rangle + \left(\frac{g_{\rm YM}}{q}\right)^2 \langle \beta\,,\, \beta\rangle + \frac{1}{2}\left\langle e\,,\, \Delta'_{A_0} e\right\rangle$$

plus the gauge-fixing terms. Therefore, we see that the $\alpha\beta$- and e-sectors decouple. Since we have both a term in $g_{\rm YM}/q$ and one in $q/g_{\rm YM}$, we must take $q \sim g_{\rm YM}$ (a convenient choice is $q = \sqrt{2}\, g_{\rm YM}$).

Perturbative expansion around a flat connection If the connection A_0 is flat, then $B_0 = F_{A_0} = 0$ and there are no terms in $q/g_{\rm YM}$ in the

quadratic part of the action. Therefore, we can also take $g_{\rm YM} \ll q \ll 1$ and consider $(g_{\rm YM}/q)^2 \langle \beta\,,\, \beta \rangle$ as a perturbation. More precisely, we take

$$i \langle \beta\,,\, *d_{A_0}\alpha \rangle + \frac{1}{2}\left\langle e\,,\, \Delta'_{A_0} e \right\rangle + \text{gauge-fixing terms} \tag{59}$$

as unperturbed action. This is possible since the quadratic form in (59) is non-degenerate. In fact, the kernel is determined by the conditions

$$d_{A_0}\beta = 0, \quad d^*_{A_0}\beta + d_{A_0}\overline{\widetilde{\phi}_2} = 0.$$

(Notice that $b^1[A_0] = 0$ if A_0 is flat.)

Applying d_{A_0} to the second equation we get $\Delta_{A_0}\beta = 0$. The kernel is therefore empty if there are no harmonic two forms (and in general is finite dimensional).

The propagators can be computed easily. The $\alpha\beta$ propagator (i.e., the inverse of $*d_{A_0}$ on its image) is the same as in pure BF theory in the covariant gauge, as is clear by comparing (59) with (101); viz., it is the integral kernel of (generalized) Gauss linking numbers.[15] The ee-propagator is the same as the propagator for the fluctuation of the connection in YM theory, as is clear by comparing (59) with (12).

The perturbative expansion will then be organized as a formal double expansion in q and $(g_{\rm YM}/q)$. Notice that the theory is however independent of q; in fact, a rescaling $q \to tq$ can be reabsorbed by the rescaling $\alpha \to \alpha/t, \beta \to t\beta, h_{\widetilde{\psi}} \to h_{\widetilde{\psi}}/t$. This reflects the analogous independency on the coupling constant found in pure BF theory in any dimension.

It is conceivable that quantization might break this symmetry if we consider B-dependent observables. (The equivalence with YM theory rules out this possibility when we consider only YM observables.)

3.2 The self-dual gauge fixing

Preliminaries The space of two-forms can canonically be split into the sum of self-dual and anti-self-dual forms [denoted by $\Omega^{(2,+)}(M, \mathrm{ad}P)$ and $\Omega^{(2,-)}(M, \mathrm{ad}P)$] which satisfy $P^+\omega = \omega$ and $P^-\omega = \omega$ respectively, where

[15] Recall that in four dimensions we have linking numbers between spheres and loops as opposed to the standard linking numbers between loops in three dimensions

the projection operators P^+ and P^- are defined as

$$P^{\pm} = \frac{1 \pm *}{2}. \qquad (60)$$

By using one of these projection operators (whatever follows is true by replacing self-duality by anti-self-duality everywhere), we can define a new operator D_A on the complex $\Omega^*(M,\,\mathrm{ad}P) \ominus \Omega^{(2,-)}(M,\,\mathrm{ad}P)$ as

$$D_A := \left\{ \begin{array}{ccccc} d_A & : & \Omega^0(M,\,\mathrm{ad}P) & \to & \Omega^1(M,\,\mathrm{ad}P) \\ \sqrt{2}P^+ d_A & : & \Omega^1(M,\,\mathrm{ad}P) & \to & \Omega^{(2,+)}(M,\,\mathrm{ad}P) \\ \sqrt{2} d_A & : & \Omega^{(2,+)}(M,\,\mathrm{ad}P) & \to & \Omega^3(M,\,\mathrm{ad}P) \\ d_A & : & \Omega^3(M,\,\mathrm{ad}P) & \to & \Omega^4(M,\,\mathrm{ad}P) \end{array} \right. \qquad (61)$$

Then we can define the elliptic operator

$$\widetilde{\Delta}_A = D_A^* D_A + D_A D_A^*, \qquad (62)$$

and prove the following identities for this deformed Laplace operator on forms of various degrees:

$$\begin{array}{rcl} \widetilde{\Delta}_A^{(0)} & = & \Delta_A^{(0)}, \\ \widetilde{\Delta}_A^{(1)} & = & \Delta_A^{(1)} - *[F_A,\ ], \\ \widetilde{\Delta}_A^{(2)} & = & 2D_A D_A^* = 2D_A^* D_A = 2P^+ \Delta_A^{(2)} P^+. \end{array} \qquad (63)$$

Since we are considering only irreducible connections, the (deformed) Laplace operator is invertible on zero forms.

We will denote by $\widetilde{\mathrm{Harm}}_A(M,\,\mathrm{ad}P)$ the (finite) kernel of $\widetilde{\Delta}_A$. Notice that

$$\begin{array}{rcl} \widetilde{\mathrm{Harm}}_A^0(M,\,\mathrm{ad}P) & = & \mathrm{Harm}_A^0(M,\,\mathrm{ad}P) = \{0\}, \\ \widetilde{\mathrm{Harm}}_A^1(M,\,\mathrm{ad}P) & \supset & \mathrm{Harm}_A^1(M,\,\mathrm{ad}P), \\ \widetilde{\mathrm{Harm}}_A^2(M,\,\mathrm{ad}P) & = & P^+ \mathrm{Harm}_A^2(M,\,\mathrm{ad}P); \end{array} \qquad (64)$$

As in the case of the ordinary covariant Laplacian, see (52), we can define the invertible operator

$$\widetilde{\Delta}'_A = \widetilde{\Delta}_A + \pi_{\widetilde{\mathrm{Harm}}_A} = \left\{ \begin{array}{ll} 1 & \text{on } \ker(\widetilde{\Delta}_A) = \widetilde{\mathrm{Harm}}_A \\ \widetilde{\Delta}_A & \text{on } \mathrm{coker}(\widetilde{\Delta}_A) \end{array} \right. \qquad (65)$$

and its inverse $\widetilde{G}_A$.

Finally, if A is a non-trivial anti-self-dual connection (i.e., $P^+F_A = 0$), then we assume that

$$D_A : \Omega^1(M, \,\mathrm{ad}P) \to \Omega^{(2,+)}(M, \,\mathrm{ad}P)$$

is surjective, or equivalently, that

$$\ker(D_A^*) = \{0\}; \quad D_A^* : \Omega^{(2,+)}(M, \,\mathrm{ad}P) \to \Omega^1(M, \,\mathrm{ad}P). \tag{66}$$

Notice that (66) is verified for a dense set of (conformal classes of) metrics for $G = SU(2)$ [14].

For any $\sigma \in \Omega^1(M, \,\mathrm{ad}P)$ we have

$$D_{A+t\sigma}^* = D_A^* + tQ_\sigma,$$

where the A-independent operator Q_σ is defined by:

$$Q_\sigma\varphi \equiv \sqrt{2} * [\sigma, \varphi]. \quad \varphi \in \Omega^{(2,+)}(M, \,\mathrm{ad}P).$$

This implies that for t sufficiently small $D_{A+t\sigma}^*$ is also invertible and that there is neighborhood of the space $\mathcal{M}^-$ of anti-self-dual connections—which we will denote by $\mathcal{N}$—where the property (66) holds. We take $\mathcal{N}$ to be the inverse image of a neighborhood of the moduli space. By (66) it follows that $\widetilde{\Delta}_A^{(2)}$ is invertible if $A \in \mathcal{N}$, so $\widetilde{\mathrm{Harm}}_A^2(M, \,\mathrm{ad}P) = \{0\}$.

If A is an anti-self-dual connection, then $\widetilde{\mathrm{Harm}}_A^1(M, \,\mathrm{ad}P) = T_A\mathcal{M}^-$; therefore, for A in a neighborhood of $\mathcal{M}^-$, $\dim \widetilde{\mathrm{Harm}}_A^1(M, \,\mathrm{ad}P) = \dim \mathcal{M}^- = m^-$. Moreover, $D_A^2 = F_A^+ = 0 : \Omega^0(M, \,\mathrm{ad}P) \to \Omega^{(2,+)}(M, \,\mathrm{ad}P)$. This implies that

$$(\mathrm{Im}\, D_A^* \bigcap \mathrm{coker}\, D_A^*) \bigcap \Omega^1(M, \,\mathrm{ad}P) = \{0\} \quad \text{if } A \in \mathcal{M}^-.$$

Therefore, (66) is an injection from $\Omega^{(2,+)}(M, \,\mathrm{ad}P)$ to $\ker D_A^* \bigcap \Omega^1(M, \,\mathrm{ad}P)$. By continuity this property will hold in a neighborhood of $\mathcal{M}^-$. We will denote by $\mathcal{N}'$ the intersection of this neighborhood with $\mathcal{N}$ and with the neighborhood where $\dim \widetilde{\mathrm{Harm}}_A^1(M, \,\mathrm{ad}P)$ is constant.

Therefore, the neighborhood $\mathcal{N}'$—which we will use in the rest of this section—is characterized by the following two properties:

1. $D_A : (\ker D_A^* \bigcap \Omega^1(M, \,\mathrm{ad}P)) \to \Omega^{(2,+)}(M, \,\mathrm{ad}P)$ is surjective if $A \in \mathcal{N}'$;

2. $\dim \widetilde{\mathrm{Harm}}_A^1(M, \,\mathrm{ad}P) = m^-$ if $A \in \mathcal{N}'$.

The definition of the self-dual gauge fixing Now we are in a position to define the self-dual gauge fixing (for further details, s. subsec. 5.6) in terms of a gauge-fixing condition on the connection $A \in \mathcal{N}'$ together with the conditions

$$D_A^* \eta = 0, \quad \eta \perp \widetilde{\mathrm{Harm}}_A^1(M, \mathrm{ad}P), \quad P^+ B = 0, \tag{67}$$

and, by consistency,

$$D_A^* \tau = 0, \quad \tau \perp \widetilde{\mathrm{Harm}}_A^1(M, \mathrm{ad}P). \tag{68}$$

In the context of BRST quantization, the last conditions will imply

$$D_A^* \widetilde{\psi} = 0, \quad \widetilde{\psi} \perp \widetilde{\mathrm{Harm}}_A^1(M, \mathrm{ad}P). \tag{69}$$

Also in this case we have gauge fixings which are interpolating between (67) and the trivial gauge fixing $\eta = 0$. In fact, the trivial gauge fixing can be written as

$$D_A^* \eta = 0, \quad \eta \perp \widetilde{\mathrm{Harm}}_A^1(M, \mathrm{ad}P), \quad D_A \eta = 0.$$

The interpolating gauge fixings can be then written as

$$\lambda P^+ B + (1 - \lambda) D_A \eta = 0,$$

with $\lambda \in [0, 1]$.

Again one might also choose λ to be smooth but not constant on $\mathcal{A}$. In particular, if we choose λ to be constant and equal to 1 in an open neighborhood of $\mathcal{M}^-$ contained in the neighborhood $\mathcal{N}'$, and constant and equal to 0 outside $\mathcal{N}'$, we obtain a gauge fixing that is defined on the whole space $\mathcal{A}$ and restricts to the self-dual gauge fixing close to the anti-self-dual connections.

3.2.1 Classical equivalence

First we observe that an anti-self-dual connection solves the YM equations of motion. Then we see that the self-dual part of the first equation of (29) reads

$$D_A \eta = 0,$$

which together with the gauge-fixing conditions implies

$$\eta = 0; \tag{70}$$

therefore, we get

$$B = -\frac{i}{2g_{\mathrm{YM}}^2} * F_A. \tag{71}$$

Notice that this solution is the same as those obtained with the trivial and the covariant gauge fixings.

3.2.2 Quantum equivalence

To implement the self-dual gauge fixing, we have to introduce a BRST complex which is slightly different from that used for the covariant case.

More precisely, we have to replace the pairs $(\overline{\widetilde{\psi}}, \overline{\widetilde{\phi}_2})$ and $(h_{\widetilde{\psi}}, h_{\widetilde{\phi}_2})$ respectively by the self-dual antighost $\overline{\chi}^+$ (with ghost number -1) and by the self-dual Lagrange multiplier h_χ^+ (with ghost number 0). Notice that the number of degrees of freedom is preserved; in fact, $\overline{\widetilde{\psi}}$ is a one-form with ghost number -1 (so four fermionic degrees of freedom), while $\overline{\widetilde{\phi}_2}$ is a zero-form with ghost number 0 (so one bosonic or, equivalently, minus one fermionic degree of freedom); this gives three fermionic degrees of freedom which is consistent with the fact that $\overline{\chi}^+$ is a self-dual two-form with ghost number -1. A similar counting holds for the other fields.

The BRST transformation rules for the antighosts and the Lagrange multipliers are the same as those described in the case of the covariant gauge fixing; as for the new fields, we have

$$s\overline{\chi}^+ = h_\chi^+, \quad sh_\chi^+ = 0. \tag{72}$$

To deal with the harmonic one-forms of the deformed Laplace operator, we introduce an orthogonal basis for $\widetilde{\mathrm{Harm}}_A^1(M, \mathrm{ad}P)$—which we still denote by $\omega_i[A]$, $i = 1, \ldots, m^-$—and normalize it as in (42).

As in the case of the covariant gauge fixing, we introduce new constant ghosts k^i and r^i, together with their antighosts and Lagrange multipliers, with BRST transformation rules given by (44) and (47). Moreover, we rewrite the BRST transformations for η and $\widetilde{\psi}$ as in (45).

The self-dual gauge fixing is eventually implemented by choosing the following gauge-fixing fermion:

$$
\begin{aligned}
\Psi \;=\; & \Psi_{\mathrm{YM}}+ \\
+ \; & \left\langle \overline{\xi}\,,\, D_A^* \eta \right\rangle + \overline{k}^i \left\langle \omega_i[A]\,,\, \eta \right\rangle + \\
+ \; & \left\langle \overline{\widetilde{\phi}_1}\,,\, D_A^* \widetilde{\psi} \right\rangle + \overline{r_1}^i \left\langle \omega_i[A]\,,\, \widetilde{\psi} \right\rangle + \\
+ \; & \left\langle \overline{\chi}^+\,,\, B \right\rangle ,
\end{aligned}
\tag{73}
$$

The canonical dimensions of the old fields are the same as in the case of the covariant gauge fixing, while the new fields $\overline{\chi}^+$ and h_χ^+ have both dimension two.

The explicit computation As in the computation with the covariant gauge fixing, the gauge-fixed action $S_{\mathrm{BFYM}} + is\Psi$ can be simplified if one imposes the gauge-fixing conditions explicitly. At the end we get

$$
\begin{aligned}
S_{\mathrm{BFYM}}^{\mathrm{s.d.\,g.f.}} \;=\; & -i \left\langle B^-\,,\, P^- F_A \right\rangle + g_{\mathrm{YM}}^2 \left\langle B^-\,,\, B^- \right\rangle + \\
+ \; & \tfrac{1}{2} \left\langle \eta\,,\, \Delta_A \eta \right\rangle - \sqrt{2} g_{\mathrm{YM}} \left\langle B^-\,,\, P^-(d_A \eta) \right\rangle + \\
+ \; & i \Big(s\Psi_{\mathrm{YM}} + \left\langle h_\xi\,,\, D_A^* \eta \right\rangle + h_k^i \left\langle \omega_i[A]\,,\, \eta \right\rangle + \\
+ \; & \left\langle h_{\widetilde{\phi}_1}\,,\, D_A^* \widetilde{\psi} \right\rangle + h_{r_1}^i \left\langle \omega_i[A]\,,\, \widetilde{\psi} \right\rangle + \left\langle h_\chi^+\,,\, B^+ \right\rangle + \\
- \; & \left\langle \overline{\xi}\,,\, \Delta_A \xi \right\rangle - \overline{k}^i k^j \delta_{ij} \sqrt{V} + \\
+ \; & \left\langle \overline{\widetilde{\phi}_1}\,,\, \Delta_A \widetilde{\phi} \right\rangle + \overline{r_1}^i r^j \delta_{ij} \sqrt{V} + \\
- \; & \tfrac{1}{\sqrt{2}\, g_{\mathrm{YM}}} \left\langle \overline{\chi}^+\,,\, P^+[F_A, \xi] \right\rangle + \left\langle \overline{\chi}^+\,,\, D_A \widetilde{\psi} \right\rangle \Big),
\end{aligned}
\tag{74}
$$

where $B^\pm$ are the self-dual and anti-self-dual components of B.

Notice that there is only one term which is singular as $g_{\mathrm{YM}} \to 0$. However, this singularity can be easily removed if one rescales $\xi \to g_{\mathrm{YM}}\, \xi$ and $\overline{\xi} \to \overline{\xi}/g_{\mathrm{YM}}$.

Now we can start integrating out the fields.

Step 1 Integrate $\overline{k}^i, k^i, \overline{r_1}^i, r^i$.

As in the case of the covariant gauge fixing, this integration gives no contribution.

Step 2 Integrate $\overline{\xi}, \xi, \overline{\widetilde{\phi}_1}, \widetilde{\phi}$.

Again, this integration does not contribute.

Step 3 Integrate $\overline{\chi}, \widetilde{\psi}, h_{\widetilde{\phi}_1}, h^i_{r_1}$.

The relevant terms in the action can be written as

$$\frac{i}{2} \langle X \,,\, MX \rangle \,,$$

where X is the vector

$$X = \begin{pmatrix} \widetilde{\psi} \\ \mathbf{h}_{r_1} \\ \overline{\chi}^+ \\ h_{\widetilde{\phi}_1} \end{pmatrix} \in \Omega^1 \oplus \mathbf{R}^{m^-} \oplus \Omega^{2,+} \oplus \Omega^0,$$

and M is the anti-hermitean operator

$$M = \begin{pmatrix} 0 & -\boldsymbol{\omega}_A & -D^*_A & -D_A \\ \boldsymbol{\omega}^*_A & 0 & 0 & 0 \\ D_A & 0 & 0 & 0 \\ D^*_A & 0 & 0 & 0 \end{pmatrix}. \tag{75}$$

The scalar product is defined as in (2) on $\Omega^*(M, \mathrm{ad}P)$ and is the ordinary Euclidean scalar product on $\mathbf{R}^{m^-}$.

The operator $\boldsymbol{\omega}_A : \mathbf{R}^{m^-} \to \Omega^1(M, \mathrm{ad}P)$ is defined by

$$\boldsymbol{\omega}_A \mathbf{h}_{r_1} = \sum_{i=1}^{m^-} \omega_i[A]\, h^i_{r_1},$$

and its adjoint $\boldsymbol{\omega}^*_A : \Omega^1(M, \mathrm{ad}P) \to \mathbf{R}^{m^-}$ acts as

$$(\boldsymbol{\omega}^*_A \widetilde{\psi})_i = \left\langle \omega_i[A] \,,\, \widetilde{\psi} \right\rangle .$$

The functional integration will then produce the Pfaffian of M which, as we will prove in App. A, is given, up to an irrelevant constant, by

$$\mathrm{Pf}(M) \propto (\det(\Delta^{(0)}_A - R_A) \det{}' \widetilde{\Delta}^{(1)}_A \, \det \Delta^{(2,+)}_A)^{1/4} \; V^{m^-/8}, \tag{76}$$

where

$$R_A = D^*_A \pi_{\mathrm{coker}(D_A)} D_A : \Omega^0(M, \mathrm{ad}P) \to \Omega^0(M, \mathrm{ad}P). \tag{77}$$

Notice that the operator

$$\widehat{\Delta}_A = \Delta_A - R_A : \Omega^0(M, \mathrm{ad}P) \to \Omega^0(M, \mathrm{ad}P) \tag{78}$$

is invertible for $A \in \mathcal{N}'$ (s. App. B).

Step 4 Integrate h^+_χ, B.

First notice that, since self-dual and anti-self-dual two-forms are orthogonal, the integration over B can be replaced by an integration over B^+ and B^- with Jacobian equal to 1.

The (h^+_χ, B^+)-integration is then trivial. The B^--integration with source

$$\left\langle B^- \, , \, P^-(-iF_A - \sqrt{2} g_{\rm YM}\, d_A\eta)\right\rangle$$

yields a constant term depending on $g_{\rm YM}$ (of which we do not care) plus the following contribution to the action

$$\frac{1}{4g^2_{\rm YM}}\left\langle P^-F_A\, , \, P^-F_A\right\rangle - \frac{i}{\sqrt{2}\, g_{\rm YM}}\left\langle P^-F_A\, , \, P^-(d_A\eta)\right\rangle - \frac{1}{2}\left\langle P^-(d_A\eta)\, , \, P^-(d_A\eta)\right\rangle .$$

Therefore, at this stage we get the following effective action:

$$\frac{1}{4g^2_{\rm YM}}\left\langle P^-F_A\, , \, P^-F_A\right\rangle + \frac{1}{4}\left\langle \eta\, , \, \widetilde{\Delta}'_A\eta\right\rangle + i\left\langle \eta\, , \, \frac{-1}{2g_{\rm YM}}\, D^*_A P^+ F_A + D_A h_\xi + h^i_k \omega_i[A]\right\rangle + is\Psi_{\rm YM},$$

where we have used the fact that $\sqrt{2} d^*_A P^- F_A = \sqrt{2} d^*_A P^+ F_A = D^*_A P^+ F_A$.

Step 5 η, h_ξ, h^i_k.

The η-integration yields $(\det{}' \widetilde{\Delta}^{(1)}_A)^{-1/2}$ plus the following contribution to the action:

$$\frac{1}{4g^2_{\rm YM}}\left\langle P^+F_A\, , \, \widetilde{Z}_A P^+F_A\right\rangle - \frac{1}{g_{\rm YM}}\left\langle h_\xi\, , \, D^*_A \widetilde{G}_A D^*_A P^+ F_A\right\rangle + \left\langle h_\xi\, , \, \widetilde{X}_A h_\xi\right\rangle + h^i_k h^j_k \delta_{ij}\sqrt{V},$$

where

$$\begin{array}{rcl} \widetilde{X}_A &=& D^*_A \widetilde{G}_A D_A \ : \ \Omega^0(M, \mathrm{ad}P) \ \rightarrow \ \Omega^0(M, \mathrm{ad}P), \\ \widetilde{Z}_A &=& D_A \widetilde{G}_A D^*_A \ : \ \Omega^{(2,+)}(M, \mathrm{ad}P) \ \rightarrow \ \Omega^{(2,+)}(M, \mathrm{ad}P). \end{array} \tag{79}$$

Even though D_A does not commute with G_A (unless $A \in \mathcal{M}^-$), these two operators are identity operators as long as $A \in \mathcal{N}'$. For details, s. App. B.

The h^i_k-integrations yield $(4V)^{-m^-/4}$, while the h_ξ-integration produces $(\det \widetilde{X}_A)^{-1/2} = 1$ plus the contribution

$$-\frac{1}{4g^2_{\rm YM}}\left\langle P^+F_A\, , \, \widehat{Z}_A P^+F_A\right\rangle ,$$

where

$$\widehat{Z}_A = D_A \widetilde{G}_A D_A D^*_A \widetilde{G}_A D^*_A : \Omega^{(2,+)}(M, \mathrm{ad}P) \rightarrow \Omega^{(2,+)}(M, \mathrm{ad}P). \tag{80}$$

As long as $A \in \mathcal{N}'$, this operator is null as proved in App. B.

Conclusions Putting together the determinants coming from steps 3 and 5 we find a net contribution

$$J[A] = \frac{(\det(\Delta_A^{(0)} - R_A))^{1/4} (\det \Delta_A^{(2,+)})^{1/4}}{(\det' \widetilde{\Delta}_A^{(1)})^{1/4}}. \tag{81}$$

In App. B, we show that $J[A] = 1$ if $A \in \mathcal{N}'$. Moreover, Step 4 and Step 5 reconstruct YM action in the form

$$S_{\mathrm{YM}}[A] = \frac{1}{4g_{\mathrm{YM}}^2} \left\langle P^- F_A \,,\, P^- F_A \right\rangle + \frac{1}{4g_{\mathrm{YM}}^2} \left\langle P^+ F_A \,,\, P^+ F_A \right\rangle .$$

Therefore, we have proved the equivalence between BFYM and YM theory (for $A \in \mathcal{N}'$) by using the self-dual gauge fixing. More explicitly, we have shown that

$$\int_{(T\mathcal{N}' \times T_{F_A}\mathcal{B})/\mathcal{G}_{\mathrm{aff}},\mathrm{self-dual}} \exp(-S_{\mathrm{BFYM}}[A, \eta, B])\, \mathcal{O}[A] \propto \int_{\mathcal{N}'/\mathcal{G}} \exp(-S_{\mathrm{YM}}[A])\, \mathcal{O}[A]. \tag{82}$$

If we choose a gauge fixing that restricts to the self-dual gauge in the interior of $\mathcal{N}'$ and to the trivial gauge outside, we can extend the equivalence to the whole $\mathcal{A}$.

3.2.3 The perturbative expansion around an anti-self-dual connection

Again we can consider fluctuations around a background as in (57). Since we assume the connection not to be flat, we will have both terms in q/g_{YM} and in g_{YM}/q, so we must take $q \sim g_{\mathrm{YM}}$. It is convenient to choose $q = \sqrt{2}\, g_{\mathrm{YM}}$.

The gauge-fixing conditions (67) on the fluctuations simply read

$$d_{A_0}^* \eta + O(g_{\mathrm{YM}}) = 0, \quad \eta \perp \widetilde{\mathrm{Harm}}{}^1_{A_0}(M, \mathrm{ad}P), \quad \beta^+ = 0.$$

The quadratic part of the gauge-fixed action (74) reads

$$\begin{aligned} &-i \langle \beta^- \,,\, d_{A_0}\alpha \rangle + \langle F_{A_0} \,,\, \alpha \wedge \alpha \rangle + \tfrac{1}{2} \langle \beta^- \,,\, \beta^- \rangle + \\ &- \langle \beta^- \,,\, d_{A_0} e \rangle - i \langle F_{A_0} \,,\, [\alpha, e] \rangle + \tfrac{1}{2} \left\langle e \,,\, \widetilde{\Delta}'_{A_0} e + *[F_{A_0}, e] \right\rangle , \end{aligned}$$

plus the gauge-fixing terms. Unlike in the case of the covariant gauge fixing, the $\alpha\beta$- and e-sectors do not decouple. However, if we perform the change of variables

$$\alpha' = \alpha - ie, \quad e' = e, \quad \beta' = \beta, \tag{83}$$

the quadratic part of the action turns out to be

$$-i\left\langle(\beta')^-\,,\,d_{A_0}\alpha'\right\rangle+\langle F_{A_0}\,,\,\alpha'\wedge\alpha'\rangle+\frac{1}{2}\left\langle(\beta')^-\,,\,(\beta')^-\right\rangle+\frac{1}{2}\left\langle e'\,,\,\widetilde{\Delta}'_{A_0}e'\right\rangle, \quad (84)$$

plus the gauge-fixing terms. Now the $\alpha'\beta'$- and e'-sectors decouple. Moreover, for the $\alpha'\beta'$-sector we recognize the propagators of the topological BF theory with a cosmological term in the self-dual gauge, see. (106), whereas the $e'e'$-propagator turns out to be same as the propagator for the fluctuation of the connection in YM theory [thanks to (13) and to the second equation of (63)].

4 The relation with the topological BF theories

In the previous section, studying perturbative BFYM theory in the covariant gauge around a flat connection or in the self-dual gauge around a non-trivial anti-self-dual connection, we have discovered that a sector of the theory corresponds to the topological BF theory (pure or, respectively, with a cosmological term) in the same gauge.

In this section we will recall the properties of the topological BF theories. The main problem with these theories is that the symmetries are described by a BRST operator that is nilpotent only on-shell. Therefore, one has to resort to the BV formalism which we briefly introduce in subsection 4.1.

We also want to discuss the relations between the BFYM and the BF theories before starting perturbation theory. In the case of the self-dual gauge, this relation simply relies on the fact that $\langle B\,,\,B\rangle=-\langle B\,,\,*B\rangle$ when B is anti-self-dual.

The case of the covariant gauge fixing with a flat background connection is however more intricate, for it is related to the limit $g_{\mathrm{YM}}\to 0$ which is ill-defined as discussed at the end of Sec. 2.4. We have already observed that in this limit the BFYM theory formally reduces to the topological pure BF theory plus a dynamical term for η, s. eqn. (28).

We have also observed that this limit is well-defined after fixing the gauge.

However, we would like this limit to be meaningful for the theory even before a gauge is chosen.

Dealing with the theory defined by (28) presents some difficulties. In fact, the term $\langle d_A\eta\,,\,d_A\eta\rangle$ has a different symmetry on shell (the $T\mathcal{G}$ action)

and off shell (only the $\mathcal{G}$ action). Of course, one has to consider the larger symmetry if one wants to quantize the theory. The on-shell symmetry for η can be made into an off-shell symmetry of the whole theory by setting

$$sB = [B, c] - d_A\widetilde{\psi} + *[d_A\eta, \xi].$$

However, now the BRST operator is nilpotent only on shell. (Notice that this is a problem affecting pure BF theory as well.)

Another way of seeing the problem is to perform the limit $g^2 \to 0$ in BFYM theory. We meet the following difficulties:

1. There is no way of getting in the limit the previous BRST transformation on B.

2. If we consider the BRST transformations as in (31), we get, in the limit, the correct on-shell symmetry for η but a divergent transformation for B.

3. If we try to avoid this problem by rescaling $\xi \to \sqrt{2}\, g_{\mathrm{YM}}\, \xi$, we get a well-defined transformation for B, but the transformation for η is correct only off shell now; this leads to contradictions when we try to quantize the theory. In fact, if we decide not to fix the gauge for η we get in trouble when the curvature vanishes; on the other hand, if we want to gauge fix it, we have to introduce the antighost $\overline{\xi}$, but then we get in trouble since the $\overline{\xi}$-dependent terms in the gauge-fixed action are killed. (Of course, if we first fix the gauge and then let $g_{\mathrm{YM}} \to 0$, we do not have any problem.)

4. If we also decide to rescale $\eta \to \sqrt{2}\, g_{\mathrm{YM}}\, \eta$, the quadratic term in η disappears from the action. This means that the symmetry on η is given, as we correctly obtain, by the whole $\mathcal{G}_{\mathrm{aff}}$ action. However, now B can be shifted by $d_A\widetilde{\psi}'$ with no relation between $\widetilde{\psi}$ and $\widetilde{\psi}'$. That is, we have to introduce new ghosts.

The solution to these problems is again in the use of the BV formalism.

4.1 The BV formalism

In the BV formalism, one considers the **Z**-graded algebra of polynomials in the *fields* $\{\Phi_i\}$ of the theory. We will denote by $\epsilon(K)$ the grading—i.e., the

ghost number—of the monomial K. As a shorthand notation, we will simply write ϵ_i for the ghost number of the generator Φ^i. Moreover, each field is given a Grassmann parity by the reduction mod 2 of the ghost number (if half-integer-spin particles are present, then their Grassmann parity is increased by one).

To each field Φ^i is then associated an antifield $\Phi_i^\dagger$ which is completely equivalent to Φ^i under all respects but the ghost number; i.e., $\Phi_i^\dagger$ is a section of the same principal or associated bundle as Φ^i and is given ghost number by

$$\epsilon(\Phi_i^\dagger) = -\epsilon_i - 1. \tag{85}$$

4.1.1 BV antibracket and Laplacian

Given two functions X and Y of the variables $\{\Phi^i, \Phi_i^\dagger\}$, one defines the BV antibracket as

$$(X\,,\,Y) := X\left\langle \frac{\overleftarrow{\delta}}{\delta\Phi_i^\dagger}\,,\,\frac{\overrightarrow{\delta}}{\delta\Phi^i}\right\rangle Y - X\left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^i}\,,\,\frac{\overrightarrow{\delta}}{\delta\Phi_i^\dagger}\right\rangle Y \tag{86}$$

and the BV Laplacian as

$$\Delta X = \sum_i (-1)^{\epsilon_i}\left\langle \frac{\overrightarrow{\delta}}{\delta\Phi_i^\dagger}\,,\,\frac{\overrightarrow{\delta}}{\delta\Phi^i}\right\rangle X. \tag{87}$$

Notice that both the antibracket and the Laplacian increase the ghost number by one.

We remark that the two previous operations are not independent: in fact, the BV antibraket can be written in terms of the BV Laplacian and of the pointwise product of functions.

4.1.2 Canonical transformations

The BV formalism is defined modulo *canonical transformations*, i.e., transformations of the fields and antifields that preserve the BV Laplacian and, consequently, the BV antibracket.

A canonical transformation can be obtained by introducing a generating functional $F(\Phi^i, \widetilde{\Phi_i^\dagger})$, with $\epsilon(F) = -1$, such that

$$\widetilde{\Phi^i} = \frac{\overrightarrow{\delta}}{\delta \widetilde{\Phi_i^\dagger}} F, \quad \Phi_i^\dagger = \frac{\overrightarrow{\delta}}{\delta \Phi^i} F. \tag{88}$$

In the BV context, there is no analogue of Liouville's theorem in classical mechanics, and, in general, the volume form is *not* preserved by canonical transformations.

Notice that rescalings of the form $\Phi^i \to \lambda^i \Phi^i$, $\Phi_i^\dagger = \Phi_i^\dagger/\lambda^i$ are canonical transformations, their generating functional being $F = \sum_i \lambda^i \left\langle \Phi^i \, , \, \widetilde{\Phi_i^\dagger} \right\rangle$.

4.1.3 The implementation of symmetries

Suppose we have an action $S[\varphi]$, where by φ we denote the classical fields (i.e., the zero-ghost-number fields that appear in the action). The study of the on-shell symmetries allows the construction of the BRST complex (i.e., the whole set of fields Φ^i) together with the BRST operator s. In many cases, this operator turns out to be nilpotent also off shell, and the BRST formalism is enough to quantize the theory. However, there are situations (e.g., in the BF theories) where this is not true. In these cases, the BV formalism provides a useful generalization of the BRST formalism.

First of all one has to look for the BV action, i.e., a functional $S^{\mathrm{BV}}[\Phi, \Phi^\dagger]$ that solves the *master equation*

$$\left(S^{\mathrm{BV}} \, , \, S^{\mathrm{BV}}\right) = 0, \tag{89}$$

and reduces to the classical action S when the antifields are turned off, viz.,

$$S^{\mathrm{BV}}[\Phi, 0] = S[\varphi]. \tag{90}$$

In particular, one looks for a *proper solution* of (89); i.e., one requires the Hessian of S^{BV} evaluated on-shell to have rank equal to the number of fields.

There is a theorem that states that, under some mild assumptions on S, there exists one and only one (up to canonical transformations) proper solution S^{BV} to the master equation (89) with boundary conditions (90).

Thanks to the master equation and to the properties of the BV antibracket, the operator σ defined by

$$\sigma X := (S^{\mathrm{BV}}, X) \tag{91}$$

turns out to be nilpotent. The boundary condition (90) then ensures that, up to possible terms in the antifields, σ acts on the fields as s.

If the BRST operator s is nilpotent also off shell, one can write the BV action as

$$S^{\mathrm{BV}}[\Phi, \Phi^\dagger] = S[\varphi] + \sum_i \left\langle s\Phi^i \,,\, \Phi_i^\dagger \right\rangle . \tag{92}$$

In this case $\sigma = s$ on all fields.

4.1.4 The BV quantization

The quantization of the theory then proceeds by fixing the gauge. This is achieved, as in the BRST formalism, by introducing a gauge-fixing fermion $\Psi[\Phi]$. Now, however, the gauge-fixed action is defined by

$$S^{\mathrm{g.f.}}[\Phi] = S^{\mathrm{BV}}[\Phi, \Phi^\dagger]\Big|_{\Phi_j^\dagger = i\frac{\delta}{\delta\Phi^j}\Psi} . \tag{93}$$

If S^{BV} has the form (92), then this procedure gives $S^{\mathrm{g.f.}} = S + is\Psi$, as in the BRST formalism.

The condition that S^{BV} should be a proper solution of the master equation makes perturbative quantization possible and—if $\Delta S = 0$—independent of small deformations of Ψ.[16] Moreover, the vacuum expectation value of a functional $\mathcal{O}(\Phi, \Phi^\dagger)$ such that $\sigma\mathcal{O} = 0$ turns out to be independent of small deformations of Ψ as well.

[16]Usually one can choose $\Delta S = 0$. However, the quantum corrections due to renormalization generally break this condition.

To deal with this case, one has to consider the quantum BV action $S_\hbar^{\mathrm{BV}}(\Phi, \Phi^\dagger)$ which satisfies the quantum master equation

$$\left(S_\hbar^{\mathrm{BV}} \,,\, S_\hbar^{\mathrm{BV}}\right) + 2\hbar\Delta S_\hbar^{\mathrm{BV}} = 0$$

and the boundary condition

$$S_0^{\mathrm{BV}}(\Phi, \Phi^\dagger) = S^{\mathrm{BV}}(\Phi, \Phi^\dagger).$$

Notice that to a BV action there might correspond no quantum BV action; in this case the theory is said to be anomalous.

4.2 Applications of the BV formalism

The theories we have considered in this paper—viz., first- and second-order YM theory and BFYM theory—have a BRST operator that closes also off shell; therefore, up to canonical transformations, they can be written as in (92). Explicitly, we have

$$\begin{aligned}
S^{\rm BV}_{\rm YM} &= S_{\rm YM} + \left\langle d_A c\,,\, A^\dagger\right\rangle - \left\langle \tfrac{1}{2}[c,c]\,,\, c^\dagger\right\rangle + \left\langle h_c\,,\, \overline{c}^\dagger\right\rangle;\\
S^{\rm BV}_{\rm YM'} &= S_{\rm YM'} + \left\langle d_A c\,,\, A^\dagger\right\rangle + \left\langle [E,c]\,,\, E^\dagger\right\rangle - \left\langle \tfrac{1}{2}[c,c]\,,\, c^\dagger\right\rangle + \left\langle h_c\,,\, \overline{c}^\dagger\right\rangle;\\
S^{\rm BV}_{\rm BFYM} &= S_{\rm BFYM} + \left\langle d_A c\,,\, A^\dagger\right\rangle + \left\langle [\eta,c] + d_A\xi - \sqrt{2}\, g_{\rm YM}\, \widetilde{\psi}\,,\, \eta^\dagger\right\rangle + \\
&+ \left\langle [B,c] + \tfrac{1}{\sqrt{2}\, g_{\rm YM}}\, [F_A,\xi] - d_A\widetilde{\psi}\,,\, B^\dagger\right\rangle + \\
&+ \left\langle -\tfrac{1}{2}[c,c]\,,\, c^\dagger\right\rangle + \left\langle -[\xi,c] + \sqrt{2}\, g_{\rm YM}\, \widetilde{\phi}\,,\, \xi^\dagger\right\rangle + \\
&+ \left\langle -[\widetilde{\psi},c] + d_A\widetilde{\phi}\,,\, \widetilde{\psi}^\dagger\right\rangle + \left\langle [\widetilde{\phi},c]\,,\, \widetilde{\phi}^\dagger\right\rangle + \textstyle\sum_i \left\langle h^i\,,\, \overline{c}_i^\dagger\right\rangle,
\end{aligned} \tag{94}$$

where, in the last line, we have denoted by h^i and $\overline{c}^i$ the Lagrange multipliers and antighosts.

The canonical transformation $\xi \to \sqrt{2}\, g_{\rm YM}\, \xi, \xi^\dagger \to \xi^\dagger/(\sqrt{2}\, g_{\rm YM})$ seems to remove all the singularities from $S^{\rm BV}_{\rm BFYM}$. However, in the $g_{\rm YM} \to 0$ limit the BV action turns out not to be proper. As a consequence, if we fix the gauge with Ψ as in (48), we do not get the kinetic term for $\overline{\xi}, \xi$ and quantization becomes impossible.

4.2.1 The pure BF theory

The pure BF theory is described by the action

$$S_{BF} = i\, \langle B\,,\, *F_A\rangle\,, \tag{95}$$

When the theory is not anomalous, fixing the gauge as in (93) with $S^{\rm BV}$ replaced by $S^{\rm BV}_\hbar$ yields a theory whose perturbative quantization is independent of small deformations of Ψ.

Moreover, the quantum master equation implies that the operator $\sigma_\hbar$ defined as

$$\sigma_\hbar = \sigma + \hbar\Delta$$

is nilpotent. Then one can show that the vacuum expectation value of a functional $\mathcal{O}_\hbar(\Phi, \Phi^\dagger)$ such that $\sigma_\hbar \mathcal{O}_\hbar = 0$ is independent of small deformations of Ψ.

and its symmetries are encoded by the following BRST transformations:

$$\begin{array}{rclrcl} sA & = & d_A c, & sc & = & -\frac{1}{2}[c,c], \\ sB & = & [B,c] - d_A\widetilde{\psi}, & & & \\ s\widetilde{\psi} & = & -[\widetilde{\psi},c] + d_A\widetilde{\phi}, & s\widetilde{\phi} & = & [\widetilde{\phi},c]. \end{array} \tag{96}$$

Notice that $s^2 \neq 0$ off shell; as a matter of fact, s^2 vanishes on all fields but on B where one gets

$$s^2 B = -[F_A, \widetilde{\phi}].$$

The BV action can be written as

$$\begin{aligned} S_{BF}^{\mathrm{BV}} \;=\; & S_{BF} + \left\langle d_A c\,,\, A^\dagger \right\rangle + \left\langle [B,c] - d_A\widetilde{\psi} - \tfrac{1}{2} * [B^\dagger, \widetilde{\phi}]\,,\, B^\dagger \right\rangle + \\ + \; & \left\langle -\tfrac{1}{2}[c,c]\,,\, c^\dagger \right\rangle + \left\langle -[\widetilde{\psi},c] + d_A\widetilde{\phi}\,,\, \widetilde{\psi}^\dagger \right\rangle + \left\langle [\widetilde{\phi},c]\,,\, \widetilde{\phi}^\dagger \right\rangle + \textstyle\sum_i \left\langle h^i\,,\, \overline{c}_i^\dagger \right\rangle , \end{aligned} \tag{97}$$

where the sum is over the same antighosts and Lagrange multipliers as in BFYM but $\overline{\xi}$ and h_ξ. It can be shown that (97) is a proper solution of the master equation.

Notice that the BV action is not linear in $B^\dagger$. This implies that the operator σ on B acts in a different way than the operator s; viz.,

$$\sigma B = [B,c] - d_A\widetilde{\psi} - *[B^\dagger, \widetilde{\phi}]. \tag{98}$$

On all other fields, $\sigma = s$. Notice that $\sigma^2 = 0$ since

$$\sigma B^\dagger = -[B^\dagger, c] - *F_A.$$

Quantization in the covariant gauge The covariant gauge fixing for pure BF theory is defined exactly as in BFYM theory (33) and is quantistically implemented by the same gauge-fixing fermion (48) (of course, forgetting of the conditions on η) using (93). After some algebra we can write the

gauge-fixed action as

$$\begin{aligned} S_{BF}^{\text{cov. g.f.}} &= i \langle B \,,\, *F_A \rangle + i \Big(s\Psi_{\text{YM}} + \\ &+ \Big\langle h_{\widetilde{\phi}_1} \,,\, d_A^* \widetilde{\psi} \Big\rangle + h_{r_1}^i \Big\langle \omega_i[A] \,,\, \widetilde{\psi} \Big\rangle + \\ &+ \Big\langle h_{\widetilde{\psi}} \,,\, d_A^* B + d_A \overline{\widetilde{\phi}_2} + \overline{r_2}^i \, \omega_i[A] \Big\rangle + \\ &+ \Big\langle h_{\widetilde{\phi}_2} \,,\, d_A^* \overline{\widetilde{\psi}} \Big\rangle + h_{r_2}^i \Big\langle \omega_i[A] \,,\, \overline{\widetilde{\psi}} \Big\rangle + \\ &+ \Big\langle \overline{\widetilde{\phi}_1} \,,\, \Delta_A \widetilde{\phi} \Big\rangle + \overline{r_1}^i r^j \delta_{ij} \sqrt{V} + \\ &+ \tfrac{1}{2} \Big\langle d_A \overline{\widetilde{\psi}} \,,\, *[d_A \overline{\widetilde{\psi}}, \widetilde{\phi}] \Big\rangle + \Big\langle \overline{\widetilde{\psi}} \,,\, \Delta'_A \widetilde{\psi} \Big\rangle \Big). \end{aligned} \tag{99}$$

As is usual in theories whose BRST operator is nilpotent only on shell, a cubic term appears in the ghost–antighost variables, viz., $\Big\langle d_A \overline{\widetilde{\psi}} \,,\, *[d_A \overline{\widetilde{\psi}}, \widetilde{\phi}] \Big\rangle$. This term is however killed by the $\overline{\widetilde{\phi}_1} \widetilde{\phi}$ integration since there are no sources in $\overline{\widetilde{\phi}_1}$. Recall that also in (53) the term $\Big\langle d_A \widetilde{\psi} \,,\, [F_A, \xi] \Big\rangle$ was irrelevant since the $\overline{\xi}\xi$ integration killed it. Therefore, the gauge-fixing terms in (53) are the same as those which appears in (99) plus the terms related to the gauge fixing on η. Explicitly, we have

$$S_{\text{BFYM}}^{\text{cov. g.f.}} = S_{BF}^{\text{cov. g.f.}} + g_{\text{YM}}^2 \langle B \,,\, B \rangle + \frac{1}{2} \langle \eta \,,\, \Delta'_A \eta \rangle + is \left(\Big\langle \overline{\xi} \,,\, d_A^* \eta \Big\rangle + \overline{k}^i \langle \omega_i[A] \,,\, \eta \rangle \right). \tag{100}$$

Perturbative expansion The equations of motion of BF theory are $F_A = d_A B = 0$. Therefore, A is a flat connection. In the covariant background, we also have $B = 0$ (if there are harmonic two-forms, we have to require B to be orthogonal to them).

If we then consider fluctuations around a critical background (i.e., $F_{A_0} = B_0 = 0$) as in (57), we get the quadratic action

$$i \langle \beta \,,\, *d_{A_0} \alpha \rangle + \text{gauge-fixing terms}, \tag{101}$$

which corresponds to the $\alpha\beta$ part of (59). Notice that (101) is completely independent of the parameter q.

4.2.2 The BF theory with a cosmological term

This theory is described by the action

$$S_{BF,\kappa} = i\,\langle B\,,\,*F_A\rangle + i\frac{\kappa}{2}\,\langle B\,,\,*B\rangle\,, \tag{102}$$

where κ is a coupling constant known as the cosmological constant. The symmetries are encoded in the following BRST transformations:

$$\begin{array}{rclrcl} sA &=& d_A c + \kappa\widetilde{\psi}, & sc &=& -\frac{1}{2}[c,c] - \kappa\widetilde{\phi}, \\ sB &=& [B,c] - d_A\widetilde{\psi}, & & & \\ s\widetilde{\psi} &=& -[\widetilde{\psi},c] + d_A\widetilde{\phi}, & s\widetilde{\phi} &=& [\widetilde{\phi},c]. \end{array} \tag{103}$$

Again, s^2 is not nilpotent off shell and its failure is given by

$$s^2 B = -[F_A + \kappa B, \widetilde{\phi}]$$

(notice that $[\widetilde{\psi},\widetilde{\psi}] = 0$). The BV action reads

$$\begin{array}{rcl} S^{\mathrm{BV}}_{BF,\kappa} &=& S_{BF,\kappa} + \left\langle d_A c + \kappa\widetilde{\psi}\,,\, A^\dagger\right\rangle + \left\langle [B,c] - d_A\widetilde{\psi} - \frac{1}{2} * [B^\dagger,\widetilde{\phi}]\,,\, B^\dagger\right\rangle + \\ &+& \left\langle -\frac{1}{2}[c,c] - \kappa\widetilde{\phi}\,,\, c^\dagger\right\rangle + \left\langle -[\widetilde{\psi},c] + d_A\widetilde{\phi}\,,\, \widetilde{\psi}^\dagger\right\rangle + \left\langle [\widetilde{\phi},c]\,,\, \widetilde{\phi}^\dagger\right\rangle + \sum_i \left\langle h^i\,,\, \overline{c}_i^\dagger\right\rangle, \end{array} \tag{104}$$

and the only field on which σ acts in a different way is still B. σB is still given by (98), and its nilpotency is ensured by

$$\sigma B^\dagger = [-B^\dagger, c] - *(F_A + \kappa B).$$

Quantization in the self-dual gauge The self-dual gauge is defined by putting $B^+ = 0$ and again is well defined in the same hypotheses of Sec. (3.2). Its quantum implementation is obtained by (93) with the gauge-fixing fermion (73) (forgetting of η). The gauge-fixed action then turns out to be

$$\begin{array}{rcl} S^{\mathrm{s.d.\,g.f.}}_{BF,\kappa} &=& -i\,\langle B^-\,,\, P^- F_A\rangle - i\frac{\kappa}{2}\,\langle B^-\,,\, B^-\rangle + \\ &+& i\Big(s_0\Psi_{\mathrm{YM}} + \kappa\left\langle\widetilde{\psi}\,,\, \frac{\delta\Psi_{\mathrm{YM}}}{\delta A}\right\rangle + \left\langle h_{\widetilde{\phi}_1}\,,\, D_A^*\widetilde{\psi}\right\rangle + h^i_{r_1}\left\langle\omega_i[A]\,,\,\widetilde{\psi}\right\rangle + \left\langle h^+_\chi\,,\, B^+\right\rangle + \\ &+& \left\langle\overline{\widetilde{\phi}_1}\,,\, \Delta_A\widetilde{\phi}\right\rangle + \overline{r_1}^i r^j\delta_{ij}\sqrt{V} + \kappa\left\langle\overline{\widetilde{\phi}_1}\,,\, *[\widetilde{\psi}, *\widetilde{\psi}]\right\rangle + \kappa\,\overline{r_1}^i\left\langle\widetilde{\psi}\,,\, \frac{\delta}{\delta A}\left\langle\omega_i[A]\,,\,\widetilde{\psi}\right\rangle\right\rangle + \\ &+& \frac{1}{2}\left\langle\overline{\chi}^+\,,\, [\overline{\chi}^+,\widetilde{\phi}]\right\rangle + \left\langle\overline{\chi}^+\,,\, D_A\widetilde{\psi}\right\rangle\Big), \end{array} \tag{105}$$

where s_0 is s at $\kappa = 0$.

Notice that $[\widetilde{\psi}, \widetilde{\psi}]$ vanishes, yet $[\widetilde{\psi}, *\widetilde{\psi}]$ does not. This means that we have a source in $\overline{\widetilde{\phi}}_1$; hence, the $\overline{\widetilde{\phi}}_1\widetilde{\phi}$-integration will produce a term which is quartic in the ghost variables (vix., $\left\langle [\overline{\chi}^+, \overline{\chi}^+]\,, *G_A * [\widetilde{\psi}, *\widetilde{\psi}]\right\rangle$).

Therefore, differently from the covariant case (100), the BFYM theory and the BF theory with a cosmological term in the self-dual gauge have quite different vertices. However, we can still relate their quadratic parts.

Perturbative expansion We consider fluctuations as in (57) with $q = \sqrt{\kappa}$, A_0 an anti-self-dual-connection and $B_0 = -F_{A_0}/\kappa$ (notice that this is a solution of the equations of motion in the self-dual gauge). In this case, the quadratic action reads

$$-i\left\langle \beta^-\,,\, d_{A_0}\alpha\right\rangle + i\left\langle F_{A_0}\,,\, \alpha\wedge\alpha\right\rangle - \frac{i}{2}\left\langle \beta^-\,,\, \beta^-\right\rangle + \text{gauge-fixing terms.}$$

By making the change of variables

$$\alpha' = e^{\frac{i\pi}{4}}\,\alpha, \quad \beta' = e^{-\frac{i\pi}{4}}\,\beta,$$

we get

$$-\,i\left\langle (\beta')^-\,,\, d_{A_0}\alpha'\right\rangle + \left\langle F_{A_0}\,,\, \alpha'\wedge\alpha'\right\rangle + \frac{1}{2}\left\langle (\beta')^-\,,\, (\beta')^-\right\rangle + \text{gauge-fixing terms,} \tag{106}$$

which is exactly the $\alpha'\beta'$ part of (84).

4.2.3 From BFYM to pure BF theory as $g_{\mathrm{YM}} \to 0$

We want to find a canonical transformation that lets the symmetries of BFYM theory become similar to those of BF theory. First we compute the action of σ on $B^\dagger$ in BFYM theory getting

$$\sigma B^\dagger = -[B^\dagger, c] - *F_A + \sqrt{2}\,g_{\mathrm{YM}}\,d_A\eta - 2g^2 B.$$

Then we see that, if we make the change of variables

$$B \to \widetilde{B} = B + \frac{1}{\sqrt{2}\,g_{\mathrm{YM}}} * [B^\dagger, \xi], \tag{107}$$

we get

$$\sigma\widetilde{B}=[\widetilde{B},c]-d_A\widetilde{\psi}-*[B^\dagger,\widetilde{\phi}]+*[d_A\eta,\xi]+[[B^\dagger,\xi],\xi]-\sqrt{2}\,g_{\mathrm{YM}}*[\widetilde{B},\xi]. \quad (108)$$

The transformation (107) can be obtained as a canonical transformation generated by

$$F(\Phi,\widetilde{\Phi^\dagger})=\sum_i\left\langle\Phi^i\,,\,\widetilde{\Phi_i^\dagger}\right\rangle+\frac{1}{2\sqrt{2}\,g_{\mathrm{YM}}}\left\langle\widetilde{B^\dagger}\,,\,*[\widetilde{B^\dagger},\xi]\right\rangle. \quad (109)$$

Notice that, on all other fields than B and on all antifields but $\xi^\dagger$, the transformation is the identity; on $\xi^\dagger$ we have

$$\xi^\dagger\to\widetilde{\xi^\dagger}=\xi^\dagger-\frac{1}{2\sqrt{2}\,g_{\mathrm{YM}}}*\left[B^\dagger,B^\dagger\right].$$

Therefore, in the following we will drop all the tildes but on $\widetilde{B}$ and $\widetilde{\xi^\dagger}$.

We can now rewrite the BV action for BFYM theory in the new variables:

$$\begin{aligned} S^{\mathrm{BV}}_{\mathrm{BFYM}} &= \widetilde{S}_{\mathrm{BFYM}}+\left\langle d_Ac\,,\,A^\dagger\right\rangle+\left\langle[\eta,c]+d_A\xi-\sqrt{2}\,g_{\mathrm{YM}}\,\widetilde{\psi}\,,\,\eta^\dagger\right\rangle+\\ &+\left\langle[\widetilde{B},c]-d_A\widetilde{\psi}-\tfrac{1}{2}*[B^\dagger,\widetilde{\phi}]+*[d_A\eta,\xi]\,,\,B^\dagger\right\rangle+\\ &+\left\langle\tfrac{1}{2}[[B^\dagger,\xi],\xi]-\sqrt{2}\,g_{\mathrm{YM}}*[\widetilde{B},\xi]\,,\,B^\dagger\right\rangle+\\ &+\left\langle-\tfrac{1}{2}[c,c]\,,\,c^\dagger\right\rangle+\left\langle-[\xi,c]+\sqrt{2}\,g_{\mathrm{YM}}\,\widetilde{\phi}\,,\,\xi^\dagger\right\rangle+\\ &+\left\langle-[\widetilde{\psi},c]+d_A\widetilde{\phi}\,,\,\widetilde{\psi}^\dagger\right\rangle+\left\langle[\widetilde{\phi},c]\,,\,\widetilde{\phi}^\dagger\right\rangle+\textstyle\sum_i\left\langle h^i\,,\,\overline{c}_i^\dagger\right\rangle, \end{aligned} \quad (110)$$

where $\widetilde{S}_{\mathrm{BFYM}}$ is the BFYM action evaluated at $\widetilde{B}$. Notice that now the BV action does not have singular terms in the limit $g_{\mathrm{YM}}^2\to 0$; moreover, the BV action is still proper in the limit.

Notice that, if we instead rescaled $\xi\to\sqrt{2}\,g_{\mathrm{YM}}\,\xi,\widetilde{\xi^\dagger}\to\widetilde{\xi^\dagger}/(\sqrt{2}\,g_{\mathrm{YM}})$—in order for the BV transformation (108) of $\widetilde{B}$ to become, in the limit $g_{\mathrm{YM}}^2\to 0$, the same as the BV transformation (98) on B in pure BF theory—we would not get a proper BV action in the limit.

The same problems would be encountered if we decided to rescale also $\eta\to\sqrt{2}\,g_{\mathrm{YM}}\,\eta$ unless we introduced the required new ghosts.

The partition function of BFYM at $g_{\rm YM}=0$ Consider the ($g_{\rm YM}=0$)-BFYM action

$$
\begin{aligned}
S^{\rm BV,0}_{\rm BFYM} &= \widetilde{S}^0_{\rm BFYM} + \left\langle d_A c\,,\, A^\dagger \right\rangle + \left\langle [\eta, c] + d_A\xi\,,\, \eta^\dagger \right\rangle + \\
&+ \left\langle [\widetilde{B}, c] - d_A\widetilde{\psi} - \tfrac{1}{2} * [B^\dagger, \widetilde{\phi}] + *[d_A\eta, \xi] + \tfrac{1}{2}[[B^\dagger, \xi], \xi]\,,\, B^\dagger \right\rangle + \\
&+ \left\langle -\tfrac{1}{2}[c,c]\,,\, c^\dagger \right\rangle + \left\langle -[\xi, c]\,,\, \xi^\dagger \right\rangle + \\
&+ \left\langle -[\widetilde{\psi}, c] + d_A\widetilde{\phi}\,,\, \widetilde{\psi}^\dagger \right\rangle + \left\langle [\widetilde{\phi}, c]\,,\, \widetilde{\phi}^\dagger \right\rangle + \textstyle\sum_i \left\langle h^i\,,\, \overline{c}_i^\dagger \right\rangle,
\end{aligned}
\tag{111}
$$

Notice that the equations of motion impose A to be flat. Therefore, to quantize theory, it is convenient to choose the covariant gauge-fixing fermion Ψ defined in (48).

After fixing the gauge, we have at our disposal the rescaling $\xi \to \epsilon\,\xi, \overline{\xi} \to \overline{\xi}/\epsilon$. Since the partition function does not depend on the parameter ϵ, we can as well let $\epsilon \to 0$. This way the $\eta, \overline{\xi}, h_\xi, \xi$ fields decouple from the others, and their contribution to the partition function turns out to be

$$
\frac{\det \Delta^{(0)}_A}{(\det' \Delta^{(1)}_A \, \det X_A)^{1/2}}.
$$

The B integration then selects the flat connections. The partition function of BF theory is the analytic torsion which is trivial in even dimension; moreover, notice that $X_A = 1$ if A is a flat connection. Therefore, we have

$$
Z_{\rm BFYM}|_{g^2_{\rm YM}=0} = \int_{A\in\mathcal{M}_0} \frac{\det \Delta^{(0)}_A}{(\det' \Delta^{(1)}_A)^{1/2}}, \tag{112}
$$

where $\mathcal{M}_0$ is the moduli space of flat connections.

Notice that YM theory in the limit $g^2_{\rm YM} \to 0$ leads to the same result.

5 Geometry

In this section we discuss the geometrical meaning of the set of fields appearing in (39) and (40) and of the BRST equations (31).[17] The situation is as follows:

[17] To simplify the notations, we will take $\sqrt{2}\, g_{\rm YM} = 1$ throughout this section.

1. In a topological gauge theory one deals with a connection c on the bundle of gauge orbits $\mathcal{A} \to \mathcal{A}/\mathcal{G}$, considers the corresponding connection $A + c$ on the G-bundle $P \times \mathcal{A}$ and obtain the BRST equations as the structure equations and Bianchi identities for the curvature of $A + c$ [4].

2. In a non-topological Yang–Mills theory one considers the fiber immersion $j_A : \mathcal{G} \to \mathcal{A}$ and the pulled-back connection $j_A^* c$. This is just the Maurer Cartan form on $\mathcal{G}$; the resulting structure equations for the curvature of $A + j_A^* c$ give the classical BRST equations[6]. It is customary to use the same symbol c also for the pulled-back connection $j_A^* c$: in Yang–Mills theory this is the ghost field.

3. In a full topological theory that includes the field $\eta \in \Omega^1(M, \mathrm{ad}P)$, one has to consider the tangent bundle $T\mathcal{A}$ where there is a (free) action of the tangent gauge group $T\mathcal{G}$. In complete analogy to point 1. above, one should consider a connection on $T\mathcal{A}$ and explicitly spell the structure equations and Bianchi identities for the corresponding connection on the TG-bundle $TP \times T\mathcal{A}$.

4. In a semi-topological theory that includes the field η, one should *not* follow the analogy of point 2. above, i.e., consider the $T\mathcal{G}$-orbit in $T\mathcal{A}$; instead one should take into account the pulled-back bundle $j_A^* T\mathcal{A}$ where $j_A : \mathcal{G} \to \mathcal{A}$ is the fiber immersion. The BRST equations will then be given as the structure equations and the Bianchi identities for a curvature on the bundle $TP \times j_A^* T\mathcal{A}$.

   In this way the connection A is allowed to move only in a given $\mathcal{G}$-orbit, as in the Yang–Mills theory, and the only symmetry that the theory requires for the field A is gauge invariance. On the contrary, in the bundle $j_A^* T\mathcal{A}$ the field η may be any element of $\Omega^1(M, \mathrm{ad}P) \approx T_A\mathcal{A}$; this means that the symmetries of the theory include the translation invariance for such an η.

   In other words, the theory is topological in one field-direction (η) and non-topological in another field-direction (A).

The last framework is the one that suits the theory described in this paper. Another way to see this is to start with the action (30) of the group $\mathcal{G}_{\mathrm{aff}}$

on pairs $(A, \eta) \in T\mathcal{A}$. Such an action does the required job: it gauge-transforms A and acts on η by translations. Unfortunately, such an action is not free, so the quotient space is not a manifold. In order to turn around this problem, one has to consider exactly the bundle $j_A^* T\mathcal{A}$ and obtain the BRST equations in the way mentioned above (point 4.) and discussed in details in the following pages.

It is exactly in the framework of point 4. discussed above that the field $B \in \Omega^2(M, \mathrm{ad}P)$ can be included in the BRST equations by keeping the BRST operator nilpotent.

Dealing with the tangent gauge group and with the tangent bundle of the space of connections means taking first-order approximations. These relations can be made clearer if we consider *paths (straight lines) of connections* on the bundle $P^I \times \mathcal{A}^I$, where $\mathcal{A}^I \equiv Map([0,1], \mathcal{A})$ and P^I is defined similarly.

Finally, in this section we are going to discuss the geometrical aspects of the gauge-fixing problems of our theory.

5.1 Tangent gauge group

Let P be any G-principal bundle over a closed oriented manifold M. The tangent bundle TG of any Lie group is a Lie group itself which is isomorphic to the semidirect product $G \times_s Lie(G)$ of the Lie group with its Lie algebra. The product of two pairs $(g, x), (h, y) \in G \times_s Lie(G)$ is defined as

$$(a, x)(b, y) \equiv (ab, \mathrm{Ad}_{b^{-1}}(x) + y). \tag{113}$$

Its Lie algebra is the semi-direct sum of two copies of $Lie(G)$ with commutator

$$[(x_1, y_1), (x_2, y_2)] \equiv ([x_1, x_2], [x_1, y_2] + [y_1, x_2]). \tag{114}$$

The tangent bundle TP is a TG-principal bundle with base space TM. The action of TG on TP (obtained as the derivative of the G-action on P) is given as follows

$$TP \times TG \ni (p, X)(g, x) \rightharpoonup (pg, (R_g)_* X, +i(x)_{pg}) \in TP, \tag{115}$$

where R_g denotes the (right) multiplication by $g \in G$ and $i(x)_p$ denotes the fundamental vector field corresponding to $x \in Lie(G)$ evaluated at $p \in P$.

A connection A on P is defined as a $Lie(G)$-valued one-form on P with special properties. First of all, we require its equivariance, viz.,

$$A_{pg}\left((R_g)_* X\right) = \mathrm{Ad}_{g^{-1}}\left(A_p(X)\right), \quad X \in T_pP.$$

Moreover, we require it to be the identity on fundamental vector fields:

$$A_p(i(x)_p) = x, \quad \forall x \in Lie(G).$$

A smooth map $\underline{p} : [a,b]^2 \subset \mathbb{R}^2 \to P$, with $\underline{p}(0,0) = p \in P$, defines an element of the double tangent

$$\left(p, \underline{p}', \dot{\underline{p}}, \frac{d\underline{p}'}{dt}\right) \in TTP,$$

where $(t,s) \in [a,b]^2$ and the prime denotes the derivative w.r.t. s, while the dot denotes the derivative w.r.t. t.

There is a canonical involution

$$\alpha : TTP \to TTP, \quad \alpha\left(p, \underline{p}', \dot{\underline{p}}, \frac{d\underline{p}'}{dt}\right) = \left(p, \dot{\underline{p}}, \underline{p}', \frac{d\dot{\underline{p}}}{ds}\right).$$

Now we consider the evaluation map

$$ev : \mathcal{A} \times TP \to Lie(G)$$

and its derivative

$$ev_* : T\mathcal{A} \times TTP \to Lie(G) \times Lie(G)$$

which has the following property

Theorem 1 *For any $(A,\eta) \in T\mathcal{A}$, the one-form on TP given by*

$$[\underline{p}] \rightharpoonup ev_*(A,\eta;\alpha_P[\underline{p}]) = \left(A_p\left(\underline{p}'\right), \quad \left.\frac{d}{dt}\right|_{t=0} A_{\underline{p}(0,t)}(\underline{p}'(0,t)) + \eta\left(\dot{\underline{p}}\right)\right) \qquad (116)$$

defines a connection on TP.

For the proof, s. [12]. In this way we can identify the tangent bundle $T\mathcal{A}$ as a subset of the space of connection on TP. It can be seen easily that it is a *proper* subset [12].

The gauge group $\mathcal{G}$ is the space of equivariant maps

$$\mathcal{G} = Map_G(P, G) \ni g \Rightarrow g(pa) = a^{-1}g(p)a, \ \forall a \in G.$$

We have the following:

Theorem 2 *The tangent gauge group $T\mathcal{G}$ is a* proper subgroup *of the group of gauge transformations for TP.*

Proof Let $(\psi, \chi) \in \mathcal{G} \times_s Lie(\mathcal{G})$. Notice that for any $(p, X) \in TP$ and $(g, x) \in G \times_s Lie(G)$ we have the equation

$$\begin{aligned}&\psi^{-1}d\psi\,[(pg, (R_g)_*X + i(x)|_{pg})] = \\ &\mathrm{Ad}_{g^{-1}}[\psi^{-1}d\psi(X)] + x - \mathrm{Ad}_{g^{-1}}\,\mathrm{Ad}_{\psi^{-1}(p)}\,\mathrm{Ad}_g x.\end{aligned}$$

This shows that the map

$$(\psi, \chi) : TP \longrightarrow G \times_s Lie(G)$$

given by

$$\Big(\psi(p), \psi^{-1}d\psi(p, X) + \chi(p)\Big) \in G \times_s Lie(G)$$

is a gauge transformation for TP. Notice that the above map is given by the derivative of the evaluation map $ev : P \times \mathcal{G} \to G$.

For any one-form $\eta \in \Omega^1(M, \mathrm{ad}P)$ and for any $(\psi, \chi) \in \mathcal{G} \times_s Lie(\mathcal{G})$, the map

$$(p, X) \rightharpoonup \Big(\psi(p), \psi^{-1}d\psi(p, X) + \chi(p) + \eta(p, X)\Big)$$

is also a gauge transformation on TP, thus showing that the inclusion $T\mathcal{G}_P \subset \mathcal{G}_{TP}$ is proper.

Q.E.D.

In the proof of the previous theorem we showed explicitly that the group $\mathcal{G}_{\mathrm{aff}}$ (the semidirect product of $T\mathcal{G}$ with the abelian group $\Omega^1(M, \mathrm{ad}P)$) is also a subgroup of $\mathcal{G}_{TP}$.

Remark 1 *From the discussion above we conclude that $T\mathcal{G}$ acts freely on $T\mathcal{A}$ and this action coincides with the restriction of the action of the gauge group of TP ($\mathcal{G}_{TP}$) on the space $\mathcal{A}_{TP}$ of connections on TP.*

Remark 2 *The group $\mathcal{G}_{\rm aff}$ acts non-freely on $T\mathcal{A}$ as in (21). The group $\mathcal{G}_{\rm aff}$ is a subgroup of $\mathcal{G}_{TP}$, but the action (21) is* not *given by the restriction of the action of $\mathcal{G}_{TP}$ on $\mathcal{A}_{TP}$.*

5.2 Paths on a principal bundle

For any manifold X we denote by X^I the space of smooths paths $Map(I, X)$ where $I = [0, 1]$. If $P(M, G)$ is a principal bundle, the group G^I acts freely on P^I and the bundle $P^I(M^I, G^I)$ is a principal bundle.

A path in $\mathcal{A} \equiv \mathcal{A}_P$ defines a connection on P^I. In this way we identify $\mathcal{A}^I$ with $\mathcal{A}_{P^I}$.

There is a natural bundle homomorphism

$$P^I \to TP, \quad p(t) \rightharpoonup (p(0), \dot{p}(0)) \tag{117}$$

which corresponds to the group homomorphism

$$G^I \to G \times_s Lie(G), \quad g(t) \rightharpoonup \Big(g(0), g^{-1}(0)\dot{g}(0)\Big)\,. \tag{118}$$

Under the homomorphisms (117) and (118), a connection $A(t)$ is sent into $\Big(A(0), \dot{A}(0)\Big) \in T\mathcal{A}$.

If we have a connection c (a.k.a. as a gauge fixing) on the bundle of gauge orbits $\mathcal{A} \to \mathcal{A}/\mathcal{G}$, then $A + c$ is a connection on the bundle

$$\frac{P \times \mathcal{A}}{\mathcal{G}} \mapsto M \times \frac{\mathcal{A}}{\mathcal{G}}. \tag{119}$$

In fact the one-form on $P \times \mathcal{A}$ given by

$$(A + c)_{(p,A)}(X, \eta) \equiv A(X)_p + c(\eta)_{(A,p)} \tag{120}$$

is a connection on $P \times \mathcal{A}$ which is $\mathcal{G}$-invariant, i.e., descends to a connection on the principal G-bundle (119).

Forms on $P \times \mathcal{A}$ have a bi-degree (k, s) where k is the order of the form on P and s is the order of the form on $\mathcal{A}$, a.k.a. the *ghost number*.

By taking the tangent bundles of (119) one obtain the bundle

$$\frac{TP \times T\mathcal{A}}{T\mathcal{G}} \mapsto TM \times \frac{T\mathcal{A}}{T\mathcal{G}}. \tag{121}$$

By considering the relevant path spaces, one has the bundle

$$\frac{P^I \times \mathcal{A}^I}{\mathcal{G}^I} \mapsto M^I \times \frac{\mathcal{A}^I}{\mathcal{G}^I}. \tag{122}$$

If $c(t)$ is a path of connections in $\mathcal{A} \to \mathcal{A}/\mathcal{G}$, and $A(t)$ is a path of connections in $\mathcal{A}$, then a connection on (122) is given by

$$A(t) + c_{A(t)}(t), \tag{123}$$

where we have explicitly represented the dependency of the connection $c(t)$ on the point $A(t) \in \mathcal{A}$.

As particular paths we can take straight lines,

$$A(t) = A + t\eta, \quad \eta \in \Omega^1(M, \, \mathrm{ad}P), \qquad c(t) = c + t\hat{c}, \tag{124}$$

where $\hat{c}$ is an assignment to each connection $A \in \mathcal{A}$ of a map $\hat{c}_A : \Omega^1(M, \, \mathrm{ad}P) \mapsto \Omega^0(M, \, \mathrm{ad}P)$ with the property of $\mathcal{G}$-equivariance,

$$\hat{c}_{A^g} \left(\mathrm{Ad}_{g^{-1}} \tau \right) = \mathrm{Ad}_{g^{-1}} \left(\hat{c}_A(\tau) \right),$$

and of tensoriality,

$$\mathrm{Im}\,(d_A) \subset \ker(\hat{c}_A).$$

In physics, $\hat{c}$ is an *infinitesimal variation of the gauge fixing.* It is convenient to rewrite the connection given by (124) as

$$A + t\eta + c_{A+t\eta} + t\hat{c}_{A+t\eta} = A + c_A + t\left(\eta + \xi_{A,\eta}\right) + \sum_{n=2}^{+\infty} t^n \xi^{(n)}_{A,\eta}. \tag{125}$$

In the previous expression we have:

1. identified the tangent bundle $T\mathcal{A}$ with $\mathcal{A} \times \Omega^1(M, \, \mathrm{ad}P)$ [forms on $\mathcal{A}$ can then be evaluated on elements of $\Omega^1(M, \, \mathrm{ad}P)$];

2. defined

$$\xi^{(n)}_{A,\eta}(\tau) \equiv \frac{1}{n!}\frac{d^n}{dt^n}\Big|_{t=0} c_{A+t\eta}(\tau) + \frac{1}{(n-1)!}\frac{d^{(n-1)}}{dt^{n-1}}\Big|_{t=0} \hat{c}_{A+t\eta}(\tau), \quad \tau \in \Omega^1(M, \mathrm{ad}P),$$

and set $\xi_{A,\eta} \equiv \xi^{(1)}_{A,\eta}$.

As will be shown in a moment, the pair $(A + c, \eta + \xi)$ can be seen as an honest connection with values in the Lie algebra of the tangent group TG.

First we notice that an infinite-dimensional version of (1) implies that the pair $(c, \hat{c})$ defines a connection on the $T\mathcal{G}$-bundle $T\mathcal{A}$. Explicitly we have:

Theorem 3 *When we identify the double tangent bundle $TT\mathcal{A}$ with $\mathcal{A} \times \Omega^1(M, \mathrm{ad}P)^{\times 3}$, then the connection on $T\mathcal{A}$ represented by $(c, \hat{c})$ is a map*

$$\mathcal{A}\times\Omega^1(M, \mathrm{ad}P)^{\times 3} \ni (A, \eta, \tau, \sigma) \rightharpoonup (c_A(\tau), \xi_{A,\eta}(\tau) + c_A(\sigma)) \in Lie(G)\oplus_s Lie(G).$$

Now we look again at the bundles (121) and (122).

Given the natural inclusions

$$P \to TP, p \rightharpoonup (p, 0), \quad P \to P^I, p \rightharpoonup [p(t) = p],$$

we establish from now on the following

Convention 1 *We will generally assume that the forms on $TP \times T\mathcal{A}$ we are going to consider are restricted to forms on $P \times T\mathcal{A}$ and that the forms on $P^I \times \mathcal{A}^I$ we are going to consider are restricted to forms on $P \times \mathcal{A}^I$.*

Moreover, we assume

Convention 2 *We consider only elements in $TT\mathcal{A} \approx \mathcal{A} \times \Omega^1(M, \mathrm{ad}P)^{\times 3}$ that have 0 as fourth component.*

We conclude that the $Lie(G)$-valued form

$$(A + c, \eta + \xi), \tag{126}$$

whose explicit expression is given by

$$(A + c, \eta + \xi)_{p;A,\eta}(X, \tau) = (A_p(X) + c_A(\tau), \eta_p(X) + \xi_{A,\eta}(\tau)),$$

with $p \in P, X \in T_pP$ and $(A, \eta, \tau) \in \mathcal{A}\times\Omega^1(M, \mathrm{ad}P)^{\times 2}$, represents a connection on the bundle (121) provided that *conventions* 1 and 2 are understood.

5.3 Curvatures

As is customary in topological (cohomological) field theories [4], the BRST equations are nothing but the structure equations and the Bianchi identities for the connections of some bundles of fields.

We start by recalling the expression of the curvature of the connection (120). It is given by

$$F_A + \psi + \phi, \tag{127}$$

where the three terms above are forms of degree $(2,0)$, $(1,1)$, $(0,2)$ in the product space $P \times \mathcal{A}$, the second number being the ghost number.

More precisely:

1. ψ is minus the projection of $\Omega^1(M, \, \mathrm{ad}P)$ on the horizontal subspace;

2. ϕ is the curvature of the connection c on the bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$.

.

The structure equations and the Bianchi identities in this case read

$$\begin{array}{lllllllll} \delta A & = & d_A c - \psi, & \delta c & = & -\frac{1}{2}[c,c] + \phi, & d_A F_A & = & 0, \\ \delta\psi & = & d_A\phi - [\psi, c], & \delta\phi & = & [\phi, c] & \delta F_A & = & [F, c] - d_A\psi, \end{array} \tag{128}$$

where we have denoted by δ the exterior derivative on $\mathcal{A}$, a.k.a. as the BRST operator. The total derivative for (k,s)-forms on $P \times \mathcal{A}$ is given by

$$d_{tot} = d + (-1)^k \delta. \tag{129}$$

The commutator of forms of bidegree (k,s) is assumed to satisfy the equation

$$[\omega_1^{(k_1,s_1)}, \omega_2^{(k_2,s_2)}] = (-1)^{k_1k_2+s_1s_2+1}[\omega_2^{(k_2,s_2)}, \omega_1^{(k_1,s_1)}].$$

In this way the total covariant derivative satisfies the same sign-rule as (129), namely it is given by $d_A + (-1)^k \delta_c$.

Equations (128) are the field equations for the topological field theory considered in [4].

Next we consider the curvature of (125). It is given by

$$F_A+\psi+\phi+t\left(d_A\eta + \tilde{\psi} + \tilde{\phi}\right)+t^2\left(\tilde{\psi}^{(2)} + \tilde{\phi}^{(2)} + \frac{1}{2}[\eta+\xi, \eta+\xi]\right)+o(t^2), \tag{130}$$

where the forms $\tilde{\psi}$ and $\tilde{\phi}$ are defined as

$$\begin{array}{lll} \tilde{\psi} & \equiv & d_A\xi + [\eta, c] - \delta\eta, \\ \tilde{\psi}^{(2)} & \equiv & d_A\xi^{(2)}, \\ \tilde{\phi} & \equiv & \delta\xi + [c, \xi]. \\ \tilde{\phi}^{(2)} & \equiv & \delta\xi^{(2)} + [c, \xi^{(2)}] = \delta_c\xi^{(2)} \end{array} \tag{131}$$

Accordingly the curvature of the connection (126) is given by the first-order term of (130), i.e., by the $Lie(G) \oplus_s Lie(G)$-valued form

$$(F + \psi + \phi, d_A\eta + \tilde{\psi} + \tilde{\phi}). \tag{132}$$

The structure equations and the Bianchi identity for (132) and (130) are the natural generalization of (128). They are spelled out explicitly in [12]. Here we are concerned with the geometrical interpretation of the BRST equations considered in Sec. 3, and this requires a restriction to the $\mathcal{G}$-fiber in the bundle $T\mathcal{A} \to T\mathcal{A}/T\mathcal{G}$.

5.4 Restriction to the $\mathcal{G}$-fiber

In order to obtain the standard BRST equation from (128), we need to consider the fiber imbedding

$$j_A : \mathcal{G} \to \mathcal{A}, \quad j_A(g) = A^g, \tag{133}$$

and the pulled-back bundle

$$P \times \mathcal{G} \to M \times \mathcal{G}. \tag{134}$$

The connection $A + c$ (120) on $P \times \mathcal{A}$ is pulled back to (134). It is customary (and unfortunately confusing) to denote the pulled back connection by the same symbol $A + c$. This means that in this case *c is just the Maurer–Cartan form on $\mathcal{G}$*. The curvature of the pulled-back connection is simply F_A, and the structure and Bianchi identities become

$$\delta A = d_A c, \quad \delta c = -\frac{1}{2}[c, c], \quad d_A F_A = 0, \quad \delta F_A = [F_A, c]. \tag{135}$$

These are the standard BRST equation for the Yang–Mills theory and the connection $A + c$ on (134) gives the geometrical interpretation of the set of fields and ghosts appearing in (6) [6].

Let us apply now a similar fiber imbedding to the bundle $TP \times T\mathcal{A}$. Two choices are possible:

1. consider the fiber imbedding of the full tangent gauge group $T\mathcal{G}$, i.e., the bundle: $TP \times j_{A,\eta}T\mathcal{G} \to TM \times j_{A,\eta}T\mathcal{G}$

2. consider only the fiber imbedding (133) and restrict the tangent bundle $T\mathcal{A}$ to the image of j_A. This means considering the pulled-back bundle

$$TP \times j_A^*(T\mathcal{A}) \to TM \times j_A^*(T\mathcal{A}). \tag{136}$$

The first alternative would lead us to dealing with the Maurer–Cartan form on $T\mathcal{G}$. But we are in fact interested in the second alternative since we want the field η to be generic and not restricted to be tangent to the $\mathcal{G}$-orbit. Hence, from now on, only the *second* alternative will be considered: this means that in the connection $(A+c, \eta+\xi)$ the form c becomes the Maurer–Cartan form on $j_A(\mathcal{G})$.

Taking always into consideration *convention* 1, the corresponding curvature becomes

$$(F_A, d_A\eta + \tilde{\psi} + \tilde{\phi}). \tag{137}$$

The Bianchi and structure equation for (137) are

$$\begin{array}{lcl}
\delta A & = & d_A c, \\
\delta\eta & = & -\tilde{\psi} + d_A\xi + [\eta, c], \\
\delta F_A & = & [F_A, c], \\
\delta(d_A\eta) & = & -d_A\tilde{\psi} + [F_A, \xi] + [d_A\eta, c], \\
\delta c & = & -\frac{1}{2}[c, c], \\
\delta\xi & = & \tilde{\phi} - [c, \xi], \\
\delta\tilde{\psi} & = & -[\tilde{\psi}, c] + d_A\tilde{\phi}, \\
\delta\tilde{\phi} & = & [\tilde{\phi}, c].
\end{array} \tag{138}$$

The connection $(A+c, \eta+\xi)$ for (136) gives the geometrical interpretation of the set of fields and ghosts appearing in (39) and (40) (but for the field B).

The analogous construction for the connection (125) implies the following steps:

1. take the map

$$\begin{array}{rcl}
j_A^*(T\mathcal{A}) \approx \mathcal{G} \times T_A\mathcal{A} & \to & \mathcal{A}^I, \\
(g, A, \eta) & \rightharpoonup & (A+t\eta)^g;
\end{array} \tag{139}$$

2. pull back the connection (125) to the bundle

$$P^I \times \mathcal{G} \times T_A\mathcal{A};$$

3. consider only *constant* paths in P^I, according to *convention* 1.

At this point the formal expression of the connection is the same as in (125), i.e.,

$$A + c + t\,(\eta + \xi) + \sum_{n=2}^{+\infty} t^n \xi^{(n)}, \tag{140}$$

but now c is the Maurer–Cartan form.

The relevant curvature is

$$F_A + t\left(d_A\eta + \tilde{\psi} + \tilde{\phi}\right) + t^2\left(\tilde{\psi}^{(2)} + \tilde{\phi}^{(2)} + \frac{1}{2}[\eta + \xi, \eta + \xi]\right) + o(t^2). \tag{141}$$

Computing the structure and Bianchi identities for (141) will give again (138) and some other transformation laws for the fields $\tilde{\psi}^{(n)}, \tilde{\phi}^{(n)}, \tilde{\xi}^{(n)}$ which we do not discuss here (s. [12]).

5.5 Including the field B

In four-dimensional BF quantum field theories, the field B behaves like a curvature but does not depend on the connection. It is then represented by an element of $\Omega^2(M, \operatorname{ad}P)$.

Now we show that such a field can be incorporated into the field equations (138). By incorporating we mean that the BRST double complex with operators (d, δ) can consistently be extended to a double complex with operators (d, s) that includes the space $\Omega^2(M, \operatorname{ad}P)$ in a such a way that:

1. s extends δ, so that $s^2 = 0$;

2. the gauge equivariance is preserved, and

3. the field equations are preserved.

We use here the same notation of section 2.; viz., we set

$$\mathcal{B} \equiv \Omega^2(M, \operatorname{ad}P),$$

and consider the tangent bundle

$$T\mathcal{B} \approx \Omega^2(M,\, \mathrm{ad}P) \times \Omega^2(M,\, \mathrm{ad}P).$$

The group $T\mathcal{G}$ acts on the cartesian product $T\mathcal{A} \times T\mathcal{B}$ as follows:

$$\begin{gathered}(A, \eta; C, E) \cdot (g, \zeta) = \\ \left(A^g,\, \mathrm{Ad}_{g^{-1}}\eta + d_{A^g}\zeta;\, \mathrm{Ad}_{g^{-1}}C,\, \mathrm{Ad}_{g^{-1}}E + [\,\mathrm{Ad}_{g^{-1}}C, \zeta]\right),\end{gathered} \tag{142}$$

yielding a principal $T\mathcal{G}$-bundle.

Since the projection $\mathcal{A} \times \mathcal{B} \mapsto \mathcal{A}$ is a morphism of $\mathcal{G}$-bundles, the connection c on $\mathcal{A}$ is also a connection on $\mathcal{A} \times \mathcal{B}$, and the connection (c, ξ) [determined by the pair $(c, \hat{c})$] on $T\mathcal{A}$ is also a connection on $T\mathcal{A} \times T\mathcal{B}$.

Moreover, $(A + c, \eta + \xi)$ is a *Lie*(TG)-valued *connection* on the bundle

$$\frac{TP \times T\mathcal{A} \times T\mathcal{B}}{T\mathcal{G}} \mapsto TM \times \frac{T\mathcal{A} \times T\mathcal{B}}{T\mathcal{G}}, \tag{143}$$

where again we intend to apply *conventions* 1 and 2.

Forms on $TP \times T\mathcal{A} \times T\mathcal{B}$ will be characterized by three indices (m, n, p) which represent the degree with respect to the three spaces TP, $T\mathcal{A}$, $T\mathcal{B}$. The middle integer n is again the ghost number.

The pair $(C, E) \in T\mathcal{B}$ is a *Lie*(TG)-valued $(2, 0, 0)$-form that is constant on $T\mathcal{A}$.

If we neglect the forms of degree (m, n, p) with $p > 0$, we find that, under the action of the total covariant derivative $d^{tot}_{(A+c,\eta+\xi)}$, the pair (C, E) is transformed into

$$\left(d_A C + [c, C], d_A E + [\eta, C] + [c, E] + [\xi, C]\right),$$

where we have used the fact that the pair (C, E) is constant on the space $T\mathcal{A}$.

Now we are ready to consider the possible extensions of the BRST operator δ to $T\mathcal{B}$ that satisfy the requirements 1, 2 and 3 above. In order to take into account requirement number 2, we have to consider covariant derivatives, so we set

$$s(C, E) \equiv ([C, c], [C, \xi] + [E, c]); \tag{144}$$

i.e., $(d_A - s)(C, E)$ coincides, up to forms of positive degree in the $T\mathcal{B}$-component, with $d^{tot}_{(A+c,\eta+\xi)}(C, E)$.

If the $(2,2,0)$ component of

$$\left(d^{tot}_{(A+c,\eta+\xi)}\right)^2(C,E) = [F_{(A+c,\eta+\xi)},(C,E)] = ([F_A,C],[F_A,E]+[d_A\eta+\widetilde{\psi}+\widetilde{\phi},C]) \tag{145}$$

is zero, then we may add to our field-equations (138) the transformations (144) and obtain a consistent BRST algebra that includes the elements $(C,E) \in T\mathcal{B}$.

It is a matter of simple calculations to check the following

Theorem 4 *A consistent BRST algebra that includes pairs $(C,E) \in T\mathcal{B}$ and extends (138) is possible only for pairs $(0,E)$ for any E.*

If we perform the change of variables

$$d_A\eta + E = B \tag{146}$$

and replace δ with s in (138), we obtain the following set of equations:

$$\begin{array}{rcl}
sA &=& d_A c,\\
s\eta &=& -\tilde{\psi} + d_A\xi + [\eta,c],\\
sF_A &=& [F_A,c],\\
sB &=& -d_A\tilde{\psi} + [F_A,\xi] + [B,c],\\
sc &=& -\frac{1}{2}[c,c],\\
s\xi &=& \tilde{\phi} - [c,\xi],\\
s\tilde{\psi} &=& -[\tilde{\psi},c] + d_A\tilde{\phi},\\
s\tilde{\phi} &=& [\tilde{\phi},c],
\end{array} \tag{147}$$

which are immediately recognized as the equations (31).

The change of variables (146) implies that B is a tangent vector in $T_{F_A}\mathcal{B}$. Accordingly its transformation under the group $T\mathcal{G}$ is as follows:

$$B\cdot(g,\zeta) = \operatorname{Ad}_{g^{-1}}B + [F_{A^g},\zeta]. \tag{148}$$

5.6 Gauge fixing and orbits

The fields of our theory are triples (A,η,B), where $(A,\eta) \in T\mathcal{A}$ and $B \in T_{F_A}\mathcal{B}$. This space of fields can be described conveniently by means of the curvature map

$$K : \mathcal{A} \to \mathcal{A}\times\mathcal{B}, \quad K(A) = (A, F_A), \tag{149}$$

which descends to a map

$$K : \frac{\mathcal{A}}{\mathcal{G}} \to \frac{\mathcal{A} \times \mathcal{B}}{\mathcal{G}}.$$

The space of fields [i.e., the set of triples (A, η, B)] coincides then with the set of elements of the pulled-back bundle

$$K^*(T\mathcal{A} \times T\mathcal{B}).$$

By taking into account the $T\mathcal{G}$-invariance, the space of orbits of the theory is given by

$$\frac{K^*(T\mathcal{A} \times T\mathcal{B})}{T\mathcal{G}} \approx \frac{K^*(H\mathcal{A} \times T\mathcal{B})}{\mathcal{G}}, \tag{150}$$

where by $H\mathcal{A}$ we denote the bundle of horizontal tangent vectors of $T\mathcal{A}$ with respect to a given connection on $\mathcal{A} \to (\mathcal{A}/\mathcal{G})$. The above diffeomorphism is induced by the linear map

$$[A, \eta, B]_{T\mathcal{G}} \rightharpoonup [A, \eta^H, B - d_A\eta^V]_{\mathcal{G}},$$

where the superscript H and V denote the horizontal and vertical component.

The general $\mathcal{G}_{\mathrm{aff}}$-invariance of the action (27) implies the following further translational invariance:

$$K^*(H\mathcal{A} \times T\mathcal{B}) \ni (A, \eta, B) \rightharpoonup (A, \eta + \tau, B - d_A\tau), \quad \tau \in H\mathcal{A}. \tag{151}$$

Relatively to this translational invariance, two different gauges are possible:

1. $\eta = 0$;
2. $B \perp d_A(H\mathcal{A})$; i.e., $d_A^* B \in \mathrm{Im}(d_A)$.

We can therefore identify the space of gauge-fixed fields as

$$\frac{K_2^* T\mathcal{B}}{\mathcal{G}} \approx \frac{\Big(H\mathcal{A} \ominus \mathrm{Harm}^1_A(M,\, \mathrm{ad}P)\Big) \oplus \hat{\mathcal{B}}}{\mathcal{G}}, \tag{152}$$

where $K_2 : \mathcal{A} \to \mathcal{B}$ denotes the second component of K, and $\hat{\mathcal{B}}$ is a vector bundle over $\mathcal{A}$ defined by:

$$\hat{\mathcal{B}}_A \equiv \{B \in \Omega^2(M,\, \mathrm{ad}P) \mid d_A^* B \in \mathrm{Im}(d_A)\}.$$

Let us finally come to the self-dual gauge. Here we consider the operator $D_A : H\mathcal{A} \to \Omega^{(2,+)}(M, \mathrm{ad}P)$ [s. (61)], and assume the following condition:

$$\mathrm{Im}(D_A) = \Omega^{(2,+)}(M, \mathrm{ad}P). \tag{153}$$

We know that (153) is satisfied when A is an anti-self-dual connection. By the same argument discussed in subsec. (3.2), we conclude that, if A is a connection such that (153) is satisfied, then there is a neighborhood of A in which the same condition is satisfied as well; so the set of connections for which (153) is satisfied is an open set.

On such an open set there is another way of fixing the translational invariance (151). If we set

$$B^+ = 0 \Leftrightarrow B \in \Omega^{(2,-)}(M, \mathrm{ad}P) \Leftrightarrow B \perp \Omega^{(2,+)}(M, \mathrm{ad}P), \tag{154}$$

then $\tau \in H\mathcal{A}$ is determined only up to elements in $\widetilde{\mathrm{Harm}}^1_A(M, \mathrm{ad}P)$. We can then conclude that the space of gauge-fixed fields can, in a neighborhood of an anti-self-dual connection, be given by

$$\frac{K_2^* T\mathcal{B}}{\mathcal{G}} \approx \frac{\left(H\mathcal{A} \ominus \widetilde{\mathrm{Harm}}^1_A(M, \mathrm{ad}P)\right) \oplus \Omega^{(2,-)}(M, \mathrm{ad}P)}{\mathcal{G}}. \tag{155}$$

6 Conclusions

In this paper we have discussed the possibility of describing YM theory in terms of a theory that shares many characteristics with the topological field theories (of the BF type).

The first step, Sec. 2, has been considering the first-order formulation of YM theory with the addition of an extra field to be gauged away. The resulting theory (27), which we call BFYM theory, shows a formal resemblance with pure BF theory as the coupling constant vanishes.

In Sec. 3, we have shown that BFYM theory is indeed equivalent to YM theory. Our proof is an explicit path-integral computation performed with three different (but equivalent) gauge fixings: the trivial, the covariant and the self-dual. The most interesting result is that perturbation theory in the last two gauges can be organized in a different way than in the second-order YM theory and explicitly shows the propagators of the topological BF theories.

In Sec. 4, after recalling some basic facts on the BV formalism, we have given a brief description of the BV quantization of the BF theories. Moreover, we have shown that BFYM theory can be formulated in a canonically equivalent way so that the limit for vanishing coupling is well-defined and yields the pure BF theory plus a covariant kinetic term for the extra field.

Finally, in Sec. 5, we have described the geometric structure of BFYM theory and have explicitly shown how to deal with the non-freedom of the action of the symmetry group on the space of fields.

We conclude by recalling that one of the reasons for considering BFYM theory (and looking for its underlying topological properties) is the possibility of introducing new observables that might realize ‘t Hooft’s picture; but this will be discussed elsewhere.

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A Computation of the Pfaffian of M

When studying BFYM theory in the self-dual gauge, we needed to compute the Pfaffian of the matrix M defined in (75). By a well known algebraic identity,

$$(\mathrm{Pf}M)^2 = \det M;$$

therefore,

$$(\mathrm{Pf}M)^4 = \det M^2.$$

An explicit computation using (65) and the last line of (63) yields

$$M^2 = -\begin{pmatrix} \widetilde{\Delta}'_A & 0 & 0 \\ 0 & \sqrt{V}\mathbf{1} & 0 \\ 0 & 0 & N \end{pmatrix},$$

with

$$N = \begin{pmatrix} \frac{1}{2}\widetilde{\Delta}_A & D_A D_A \\ D_A^* D_A^* & \widetilde{\Delta}_A \end{pmatrix}$$

acting on $\Omega^{(2,+)}(M, \mathrm{ad}P) \otimes \Omega^0(M, \mathrm{ad}P)$. Up to a possible irrelevant phase, the determinant of M^2 is equal to the product $V^{m^-/2} \det' \widetilde{\Delta}_A^{(1)} \det N$. An explicit computation yields

$$\det N = \det(\widetilde{\Delta}_A^{(2)}/2) \det(\widetilde{\Delta}_A^{(0)} - R_A),$$

with

$$R_A = D_A^* D_A^* 2\widetilde{G}_A D_A D_A : \Omega^0(M, \mathrm{ad}P) \to \Omega^0(M, \mathrm{ad}P).$$

To simplify R_A, we notice that

$$2\widetilde{G}_A D_A : \Omega^1(M, \mathrm{ad}P) \to \Omega^{(2,+)}(M, \mathrm{ad}P)$$

is the inverse of D_A^* on its image; more precisely, we have

$$D_A^* 2\widetilde{G}_A D_A = \pi_{\mathrm{coker}(D_A)},$$

which implies (77).

Moreover, by the last line of (63), we have

$$\det(\widetilde{\Delta}_A^{(2)}/2) = \det(\Delta_A^{(2,+)}).$$

Putting together all the pieces, we finally get (76).

B Some useful properties of the operator D_A

In Sec. 3.2, we have introduced the operator D_A. This operator is an injection from the zero- to the one-forms since A is irreducible. Moreover, we have supposed to work in a neighborhood $\mathcal{N}'$ of the space of anti-self-dual connections where D_A is also a surjection from the one- to the self-dual two-forms.

These two properties are enough to prove a series of facts which were used in Sec. 3.2 to prove the equivalence between YM theory and BFYM theory in the self-dual gauge.

In subsection B.1 we will prove these facts in general; in subsection B.2 we will specialize to our case.

B.1 The general case

Let us consider three (finite-dimensional) vector spaces X, Y and Z together with an injective linear operator

$$p : X \to Y \tag{156}$$

and a surjective linear operator

$$q : Y \to Z. \tag{157}$$

Moreover, we assume $\ker q \bigcap \ker p^* = \{0\}$.

Then we introduce the Laplace operators

$$\begin{array}{rcl} \Delta_X & = & p^* p, \\ \Delta_Y & = & pp^* + q^* q, \\ \Delta_Z & = & qq^*. \end{array} \tag{158}$$

With our hypotheses, these operators are invertible; we will denote by G_* their inverse.

Now we have the following:

Theorem 5 *If $qp = 0$, then*

1. $0 \to X \xrightarrow{p} Y \xrightarrow{q} Z \to 0$ *is an exact sequence, and*
2. $$\frac{\det \Delta_X \ \det \Delta_Z}{\det \Delta_Y} = 1.$$

Proof Fact 1 just follows from the definition of exact sequence. For fact 2, write

$$Y = Y_1 \oplus Y_2,$$

with

$$Y_1 = \operatorname{Im} p = \ker q, \quad Y_2 = \operatorname{Im} q^* = \operatorname{coker} q.$$

Δ_Y is block diagonal with respect to this decomposition, so $\det \Delta_Y = \det \Delta_{Y_1} \ \det \Delta_{Y_2}$, with

$$\begin{array}{rcl} \Delta_{Y_1} & = & pp^* \ : \ Y_1 \to Y_1, \\ \Delta_{Y_2} & = & q^* q \ : \ Y_2 \to Y_2. \end{array}$$

Moreover, Y_1 (Y_2) is isomorphic to X (Z), and p and p^* (q^* and q) are invertible operators when restricted to this space. It follows that

$$\det \Delta_{Y_1} = \det \Delta_X, \quad \det \Delta_{Y_2} = \det \Delta_Z.$$

Q.E.D.

If $pq \neq 0$, we cannot identify $\ker q$ with $\operatorname{Im} p$. We can however reduce to the preceding situation by defining

$$\bar{p} = Hp : X \to Y, \tag{159}$$

where H is the projection operator

$$H = 1 - q^* G_Z q : Y \to Y. \tag{160}$$

Since $qH = 0$, we get $q\bar{p} = 0$. We will then define

$$\begin{array}{rcl} \bar{\Delta}_X & = & \bar{p}^* \bar{p}, \\ \bar{\Delta}_Y & = & \bar{p}\bar{p}^* + q^* q. \end{array} \tag{161}$$

Then we have the following

Corollary 1 *If $\bar{p}$ is injective and q is surjective, then*

$$\frac{\det \bar{\Delta}_X \, \det \Delta_Z}{\det \bar{\Delta}_Y} = 1.$$

This corollary can however be refined to give the following

Theorem 6 *If $\bar{\Delta}_X$ is invertible and q is surjective, then*

$$\frac{\det \bar{\Delta}_X \, \det \Delta_Z}{\det \Delta_Y} = 1.$$

Proof First of all we notice that "$\bar{p}$ injective" is equivalent to "$\bar{\Delta}_X$ invertible."

Then we proceed as in the proof of Thm. 5 and split Y as

$$Y = Y_1 \oplus Y_2,$$

where $\bar{Y}_1 = \ker q$ and $Y_2 = \operatorname{coker} q$. Notice that $HY_2 = \{0\}$ and $H|_{Y_1} = 1$, so $\ker H = Y_2$.

The Laplace operator $\bar{\Delta}_Y$ is block diagonal with respect to the above decomposition of Y. Therefore,

$$\det \bar{\Delta}_Y = \det{}_{Y_1}(pp^*) \det{}_{Y_2}(q^*q).$$

The Laplace operator Δ_Y is not block diagonal but has the following form with respect to the above decomposition:

$$\Delta_Y = \begin{pmatrix} pp^* & pp^* \\ pp^* & pp^* + q^*q \end{pmatrix}.$$

By subtracting the first from the second row, we get

$$\det \Delta_Y = \det \begin{pmatrix} pp^* & pp^* \\ 0 & q^*q \end{pmatrix} = \det \bar{\Delta}_Y.$$

Q.E.D.

Remark The condition that $\bar{p}$ is injective is equivalent to $\operatorname{Im} p \bigcap \operatorname{coker} q = \{0\}$ (supposing that p is injective). If $pq = 0$, then this condition is immediate since in this case $\operatorname{Im} p = \ker q$.

A dual way of solving the problem is to define

$$\tilde{q} = qK : Y \to Z \tag{162}$$

with

$$K = 1 - pG_x p^* : Y \to Y. \tag{163}$$

In this case, we get $\tilde{q}p = 0$. We can then define $\tilde{\Delta}_Z = \tilde{q}\tilde{q}^*$ and have a theorem analogous to Thm. 6.

We summarize the previous results, and some more, in the following

Theorem 7 *If p is injective and q is surjective, then*

1. *$\bar{\Delta}_X$ is invertible if and only if $\tilde{\Delta}_Z$ is, or, equivalently, if and only if $\mathrm{Im}\, p \bigcap \mathrm{coker}\, q = \mathrm{Im}\, q^* \bigcap \mathrm{coker}\, p^* = \{0\}$.*

2. *If any one is invertible then*

   a.
$$\frac{\det \bar{\Delta}_X \; \det \Delta_Z}{\det \Delta_Y} = \frac{\det \Delta_X \; \det \tilde{\Delta}_Z}{\det \Delta_Y} = 1;$$

   b. the operators
$$\begin{array}{rcl} \xi & = & p^* G_Y p \quad : \quad X \;\to\; X, \\ \zeta & = & q G_Y q^* \quad : \quad Z \;\to\; Z \end{array}$$

   are identity operators;

   c. the operators
$$\begin{array}{rcl} \hat{\xi} & = & p^* G_Y q^* q G_Y p \quad : \quad X \;\to\; X, \\ \hat{\zeta} & = & q G_Y p p^* G_Y q^* \quad : \quad Z \;\to\; Z \end{array}$$

   are null operators.

Notice that if $pq = 0$ statements 2*b*. and 2*c*. follow trivially from the commutativity of p and q with the Laplace operators. The remarkable fact is that $\xi = \zeta = 1$ and $\hat{\xi} = \hat{\zeta} = 0$ even without this condition.

Proof To prove 1. we notice that $\tilde{\Delta}_Z$ is invertible if and only if $\tilde{q}^*$ is injective, i.e., if and only $\mathrm{Im}\, q^* \bigcap \mathrm{coker}\, p^* = \{0\}$ (supposing q^* injective). Since $\mathrm{Im}\, p = \mathrm{coker}\, p^*$ and $\mathrm{Im}\, q^* = \mathrm{coker}\, q$, the last condition turns out to be $\mathrm{Im}\, p \bigcap \mathrm{coker}\, q = \{0\}$ which is equivalent to the invertibility condition for $\bar{\Delta}_X$ (s. the preceding remark).

Statement 2*a*. follows from Thm. 6 after exchanging X with Z and p with q^*. By the way, either numerator is equal to the determinant of the operator
$$\begin{pmatrix} \Delta_Z & qp \\ p^* q^* & \Delta_X \end{pmatrix} : Z \otimes X \to Z \otimes X.$$

To prove 2b., first of all we notice that we have the commutation rules:

$$\begin{array}{rcl} p\,\Delta_X - \Delta_Y\,q & = & -q^*qp, \\ q\,\Delta_Y - \Delta_Z\,q & = & qpp^*, \end{array}$$

together with their conjugates

$$\begin{array}{rcl} \Delta_X\,p^* - p^*\,\Delta_Y & = & -p^*q^*q, \\ \Delta_Y\,q^* - q^*\,\Delta_Z & = & pp^*q^*. \end{array}$$

Now we multiply the above relations by G_* both from the left and from the right getting

$$\begin{array}{rcl} p\,G_X - G_Y\,q & = & G_Yq^*qpG_X, \\ q\,G_Y - G_Z\,q & = & -G_Zqpp^*G_Y, \\ G_X\,p^* - p^*\,G_Y & = & G_Xp^*q^*qG_Y, \\ G_Y\,q^* - q^*\,G_Z & = & -G_Ypp^*q^*G_Z. \end{array} \tag{164}$$

Using the third relation of (164), we can write

$$\xi = 1 - G_Xp^*q^*qG_Yp.$$

Then we use the second relation of (164) and obtain

$$\xi = 1 - G_XR + G_XR\xi,$$

where

$$R = p^*q^*G_Zqp.$$

Finally, we notice that

$$\bar{\Delta}_X = \Delta_X - R.$$

Applying Δ_X to the last formula for ξ, we get

$$\bar{\Delta}_X\xi = \bar{\Delta}_X.$$

If $\bar{\Delta}_X$ is invertible, this implies $\xi = 1$.

A dual proof shows that $\zeta = 1$ when $\tilde{\Delta}_Z$ is invertible.

To prove 2c., we use the fourth relation of (164) and obtain

$$\hat{\xi} = (1-\xi)p^*q^*G_ZqG_Yp = 0.$$

A dual proof shows that $\hat{\zeta} = 0$.

Q.E.D.

The infinite-dimensional case If X, Y and Z are (infinite-dimensional) Hilbert spaces, all the previous theorems hold if we add the hypothesis that p and q have elliptic Laplacians. In this case, by Hodge's theorem, the spectra of the Laplace operators are discrete, the eigenspaces are finite-dimensional and each Hilbert space is the direct sum of these eigenspaces. Therefore, we can use the ζ-function regularization. Namely, if we denote by λ the eigenvalues of a Laplace operator Δ and by d_λ the dimension of the corresponding eigenspace, the ζ-function is defined as

$$\zeta_s(\Delta) = \operatorname{Tr} \Delta^{-s} = \sum_\lambda \lambda^{-s} d_\lambda$$

for s large enough and then analytically extended. The regularized determinant is then defined as

$$\det \Delta := \exp[-\zeta_0'(\Delta)].$$

The proof of Thm. 5 can be refined by showing that p is an isomorphism between each eigenspace of Δ_X and each eigenspace of Δ_{Y_1}. This is essentially due to the fact that $p\Delta_X = \Delta_{Y_1} p$ and to the assumption that there are no zero eigenvalues. Therefore, $\zeta_s(\Delta_X) = \zeta_s(\Delta_{Y_1})$ for all s. By similar considerations on Δ_{Y_2} and Δ_Z, we get finally

$$\zeta_s(\Delta_X) - \zeta_s(\Delta_Y) + \zeta_s(\Delta_Z) = 0, \quad \forall s.$$

Deriving with respect to s and setting $s = 0$ yields (the logarithm of) the required formula.

B.2 Our case

To apply the previous analysis to our case, we set

$$\begin{aligned} X &= \Omega^0(M, \mathrm{ad}P), \\ Y &= \Omega^1(M, \mathrm{ad}P) \ominus \widetilde{\mathrm{Harm}}{}^1_A(M, \mathrm{ad}P), \\ Z &= \Omega^{(2,+)}(M, \mathrm{ad}P). \end{aligned}$$

The operators p and q correspond to the operator D_A. With these definitions we have

$$\begin{aligned} \Delta_X &= \Delta_A^{(0)}, \\ \Delta_Y &= \widetilde{\Delta}_A^{(1)}, \\ \Delta_Z &= \Delta_A^{(2,+)}, \end{aligned}$$

Moreover,

$$
\begin{array}{lcl}
\bar{\Delta}_X & = & \widehat{\Delta}_A^{(0)}, \\
\xi & = & \widetilde{X}_A, \\
\zeta & = & \widetilde{Z}_A, \\
\hat{\zeta} & = & \widehat{Z}_A,
\end{array}
$$

where the operators on the r.h.s. are defined in (78), (79) and (80).

Recall that, by the definition of $\mathcal{N}'$, $(\mathrm{Im}\, D_A^* \bigcap \mathrm{coker}\, D_A^*) \bigcap \Omega^1(M, \mathrm{ad}P) = \{0\}$ if $A \in \mathcal{N}'$. Therefore, by statement 1. of Thm. 7, $\widehat{\Delta}_A^{(0)}$ is invertible. Then, by statements 2*b*. and 2*c*. of Thm. 7 we see that

$$\widetilde{X}_A = 1, \quad \widetilde{Z}_A = 1, \quad \widehat{Z}_A = 0; \tag{165}$$

finally, by statement 2*a*., we have

$$J[A] = 1, \tag{166}$$

with $J[A]$ defined in (81).

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